Adam Blank — Spring 2016

CSE 311

Foundations of Computing I

Pre-Lecture Problem

Translate the following into predicate logic:

"There is a student who is friends with every other student except her enemies."

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no largest integer"

$$\forall x\, \left(\neg(\forall y(x \geq y))\right) \equiv \forall x\left(\exists y(\neg(x \geq y))\right)$$
$$\equiv \forall x\, (\exists y\, (x < y))$$

"For every integer there is a larger integer"

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists$ y Greater (y, x)
$\equiv \exists$ z Greater (z, x)

Truth Value:

- Doesn't depend on y or z "**bound** variables"
- Does depend on x "**free** variable"

Quantifiers only act on free variables of the formulas they quantify, e.g. $\forall$x($\exists$y (P(x,y) $\rightarrow$ $\forall$ x Q(y, x)))

Scope of Quantifiers

$\exists$x (P(x) $\wedge$ Q(x)) **vs.** $\exists$x P(x) $\wedge$ $\exists$x Q(x)

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

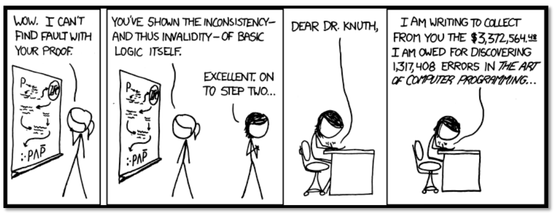
Quantifier "Style"

$\forall$x($\exists$y (P(x,y) $\rightarrow$ $\forall$x Q(y, x)))

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

CSE 311: Foundations of Computing

Lecture 6: Predicate Logic, Logical Inference



MMM Candy!

Let B be the "starred" bag of M&Ms.

Translate "There is a green M&M in B." into predicate logic.

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."

$$\text{Has}(\textbf{Green}, \textbf{B})$$

Notice that both "Green" and "B" are constants here! You could, instead, define a predicate Green(x):

$$\exists c(\text{Color}(c) \land \text{Green}(c) \land \text{Has}(c, \textbf{B}))$$

MMM Candy!

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."

Translate "Every bag of M&Ms has an M&M of some color."

$$\forall b\left(\text{Bag}(b) \to \exists c(\text{Color}(c) \land \text{Has}(b, c))\right)$$

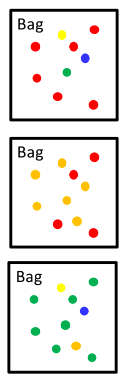
Translate "There is a color that all bags of M&Ms have."

$$\exists c\left(\text{Color}(c) \land \forall b(\text{Bag}(b) \to \text{Has}(b, c))\right)$$

We're not done!

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."



"There is a color that all bags of M&Ms have."

$$\exists c\left(\text{Color}(c) \land \forall b(\text{Bag}(b) \to \text{Has}(b, c))\right)$$

For the bags on the left, this is not true. We need **a single color** that all the bags share.

vs.

"Every bag of M&Ms has an M&M of some color."

$$\forall b\left(\text{Bag}(b) \to \exists c(\text{Color}(c) \land \text{Has}(b, c))\right)$$

For the bags on the left, this **is true**. The first bag has red; the second has orange, and the third has orange.

Still not done!

Domain of Discourse
Integers
OR
{1, 2, 3, 4}

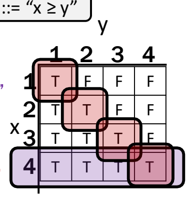
Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\left(\forall y(\text{GreaterEq}(x, y))\right)$$

"Every number has a number greater than or equal to it."

$$\forall y\left(\exists x(\text{GreaterEq}(x, y))\right)$$



The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Nested Quantifiers

- **Bound variable names don't matter**
  ∀x ∃y P(x, y) ≡ ∀a ∃b P(a, b)
- **Positions of quantifiers can sometimes change**
  ∀x (Q(x) ∧ ∃y P(x, y)) ≡ ∀x ∃y (Q(x) ∧ P(x, y))
- **But: order is important...**

Quantification with Two Variables

expression	when **true**	when **false**
∀x ∀ y P(x, y)	Every pair is true.	At least one pair is false.
∃ x ∃ y P(x, y)	At least one pair is true.	All pairs are false.
∀ x ∃ y P(x, y)	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
∃ y ∀ x P(x, y)	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Why Proofs?

Consider f(n) = 991n^2 + 1

Is f(n) a perfect square for any n > 0?

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p → q are both true then q must be true
- Write this rule as $\dfrac{p,\ p \rightarrow q}{\therefore\ q}$
- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by modus ponens:
  - You have a 311 class today.

Proofs

Show that r follows from p, p → q, and q → r

1.	p	given
2.	p → q	given
3.	q → r	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4