CSE 311

Foundations of Computing I

Pre-Lecture Problem

Translate the following into predicate logic:

"There is a student who is friends with every other student except her enemies."

yay!

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no largest integer"

$$\forall x\ \left(\neg\left(\forall y(x \geq y)\right)\right) \equiv \forall x\left(\exists y\left(\neg(x \geq y)\right)\right)$$
$$\equiv \forall x\left(\exists y\ (x < y)\right)$$

"For every integer there is a larger integer"

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists y\ \text{Greater}(y, x)$

$\equiv \exists z\ \text{Greater}(z, x)$

Truth Value:

- **Doesn't depend on y or z "bound variables"**
- **Does depend on x "free variable"**

Quantifiers only act on free variables of the formulas they quantify, e.g. $\forall x\, (\exists y\, (P(x, y) \rightarrow \forall x\, Q(y, x)))$

Scope of Quantifiers

$\exists x\, (P(x) \wedge Q(x))$ **vs.** $\exists x\, P(x) \wedge \exists z\, Q(z)$

$\exists x\, Q(x) \equiv \exists z\, Q(z)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

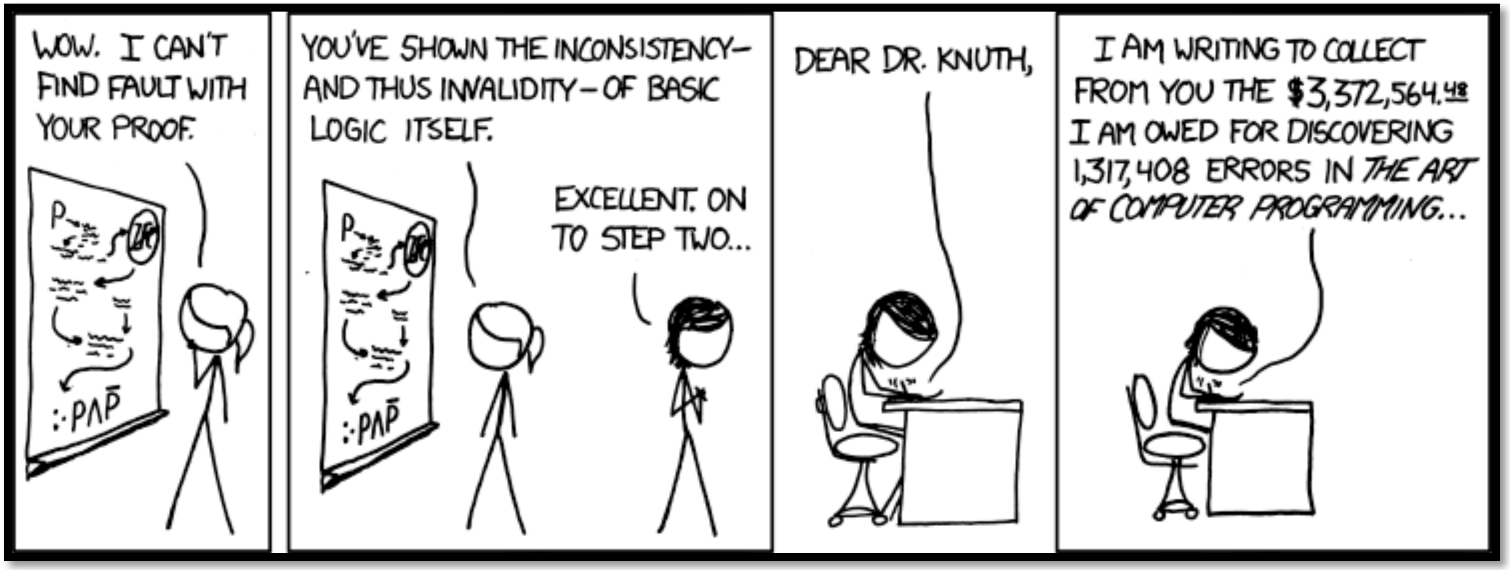
Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

CSE 311: Foundations of Computing

Lecture 6: Predicate Logic, Logical Inference



MMM Candy!

Let B be the “starred” bag of M&Ms.

Translate “There is a green M&M in B.” into predicate logic.

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Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."

Has(B, Green)

Green(x) ::= "x is Green"

$$\exists c\ ((Color(c) \land Has(B, c)) \land Green(c))$$

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$$\text{Has}(\textbf{Green}, \textbf{B})$$

Notice that both "Green" and "B" are constants here! You could, instead, define a predicate Green(x):

$$\exists \text{c}\big(\text{Color}(c) \land \text{Green}(c) \land \text{Has}(c, \textbf{B})\big)$$

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Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."

Translate "Every bag of M&Ms has an M&M of some color."

$$\forall b\, \exists c\, (Bag(b) \wedge Color(c) \wedge Has(b,c))$$

Translate "There is a color that all bags of M&Ms have."

$$\forall b\, (Bag(b) \rightarrow \exists c\, (Color(c) \wedge Has(b,c)))$$

$$\exists c\, ((Color(c) \wedge \forall b\, (Bag(b) \rightarrow Has(b,c)))$$

MMM Candy!

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Translate “Every bag of M&Ms has an M&M of some color.”

$$\forall b\left(\text{Bag}(b) \rightarrow \exists c\left(\text{Color}(c) \land \text{Has}(b, c)\right)\right)$$

Translate “There is a color that all bags of M&Ms have.”

$$\exists c\left(\text{Color}(c) \land \forall b\left(\text{Bag}(b) \rightarrow \text{Has}(b, c)\right)\right)$$

We're not done!

Domain of Discourse
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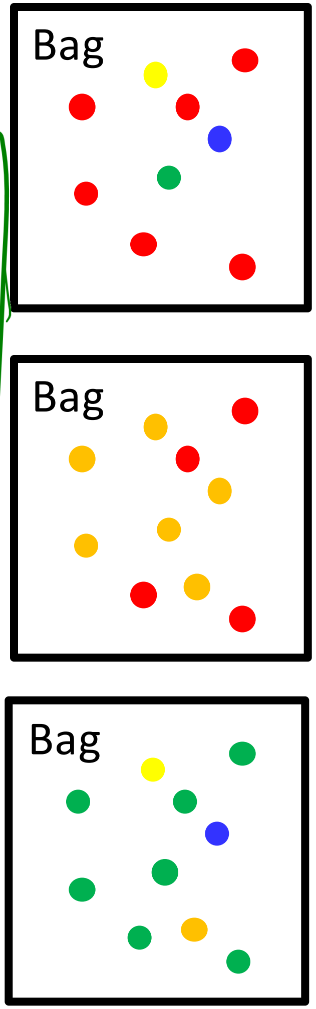

"There is a color that all bags of M&Ms have."

$$\exists c\left(\text{Color}(c) \land \forall b\left(\text{Bag}(b) \rightarrow \text{Has}(b, c)\right)\right)$$

vs.

"Every bag of M&Ms has an M&M of some color."

$$\forall b\left(\text{Bag}(b) \rightarrow \exists c\left(\text{Color}(c) \land \text{Has}(b, c)\right)\right)$$

We're not done!

Domain of Discourse
Colors & Bags

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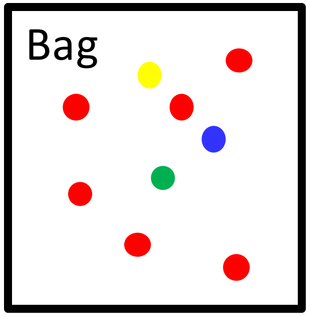
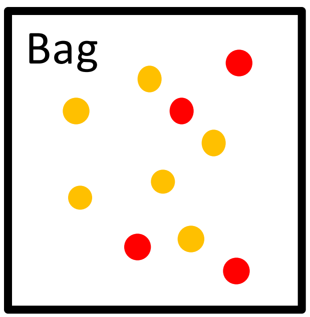
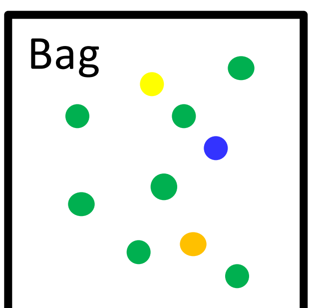






"There is a color that all bags of M&Ms have."

$$\exists c\left(\mathrm{Color}(c) \land \forall b\big(\mathrm{Bag}(b) \rightarrow \mathrm{Has}(b, c)\big)\right)$$

For the bags on the left, this is not true. We need **a single color** that all the bags share.

vs.

"Every bag of M&Ms has an M&M of some color."

$$\forall b\left(\mathrm{Bag}(b) \rightarrow \exists c\big(\mathrm{Color}(c) \land \mathrm{Has}(b, c)\big)\right)$$

For the bags on the left, this **is true**. The first bag has red; the second has orange, and the third has orange.

Still not done!

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\left(\forall y(\mathrm{GreaterEq}(x, y))\right)$$

vs.

"Every number has a number greater than or equal to it."

$$\forall y\left(\exists x(\mathrm{GreateEq}(x, y))\right)$$

Still not done!

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$$\exists x\left(\forall y\left(\text{GreaterEq}(x,y)\right)\right)$$

“Every number has a number greater than or equal to it.”

$$\forall y\left(\exists x\left(\text{GreaterEq}(x,y)\right)\right)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \land \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \land P(x, y))$$

- **But: order is important…**

Quantification with Two Variables

expression	when **true**	when **false**
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Why Proofs?

Consider $f(n) = 991n^2 + 1$

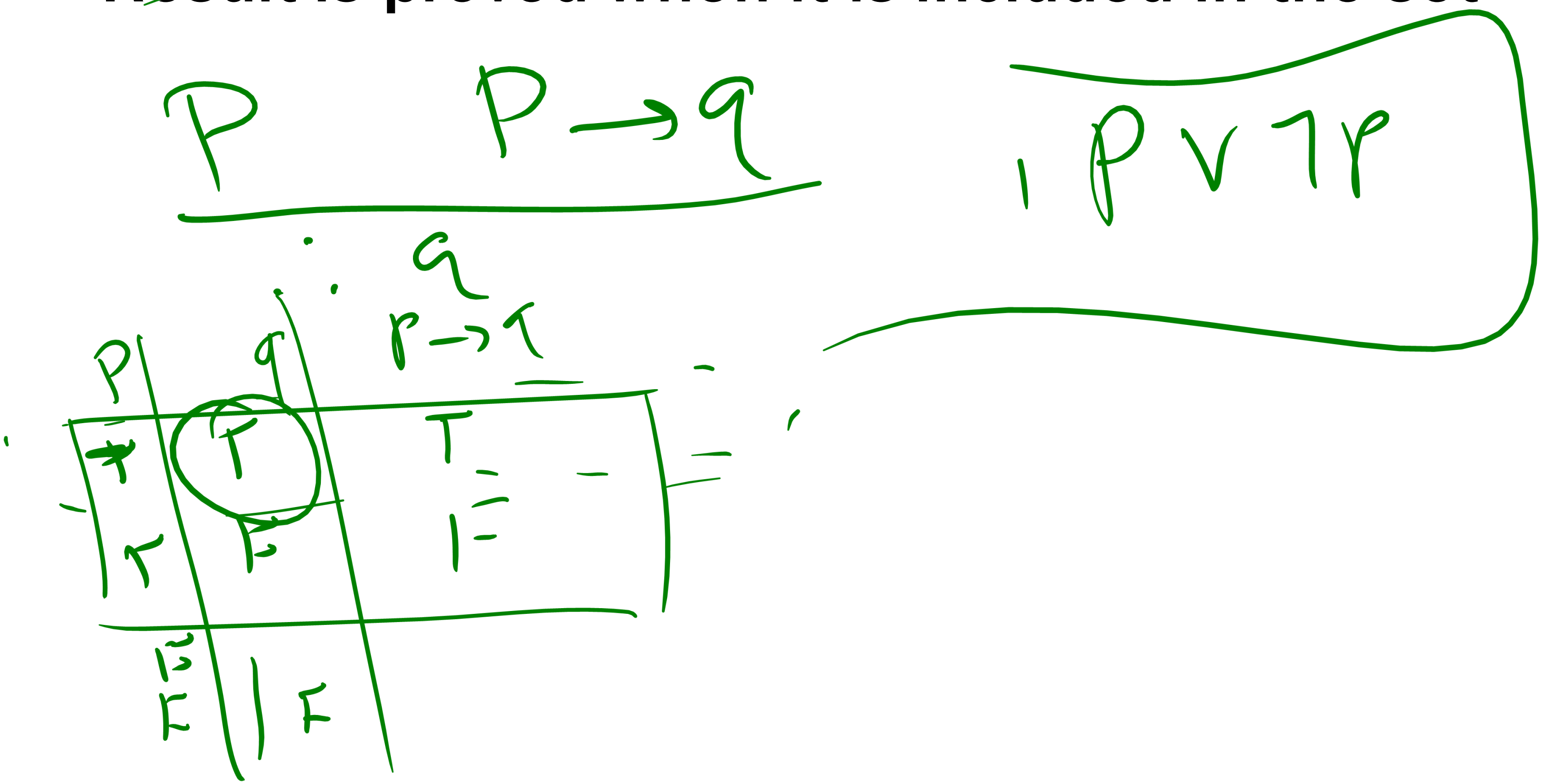
Is $f(n)$ a perfect square for any $n > 0$?

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set



An inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true

- Write this rule as

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.

- Therefore, by modus ponens:
  - You have a 311 class today.

Proofs

Show that r follows from p, p → q, and q → r

1. p given
2. p → q given
3. q → r given
4. q MP: 1,2
5. r MP: 3,4

Proofs

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	given
2.	$p \rightarrow q$	given
3.	$q \rightarrow r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4