[proved on page 86 of Volume II of Russell and Whitehead’s “Principia Mathematica”: “The above proposition is occasionally useful.”]
A set $S$ is **countable** iff we can order the elements of $S$ as $S = \{x_1, x_2, x_3, \ldots \}$

**Countable sets:**
- $\mathbb{N}$ - the natural numbers
- $\mathbb{Z}$ - the integers
- $\mathbb{Q}$ - the rationals
- $\Sigma^*$ - the strings over any finite $\Sigma$
- The set of all Java programs

*Shown by “dovetailing”*
Not every set is countable

Theorem [Cantor]:
The set of real numbers between 0 and 1 is not countable.

Proof will be by contradiction. Using a new method called diagonalization.
Real numbers between 0 and 1: [0,1)

Every number between 0 and 1 has an infinite decimal expansion:

\[ \frac{1}{2} = 0.50000000000000000000000... \quad 10 \times x = 1.999999... \]

\[ \frac{1}{3} = 0.33333333333333333333333... \quad 10 \times x = 1.999999... \]

\[ \frac{1}{7} = 0.14285714285714285714285... \quad 9 \times x = 1.800000... \]

\[ \pi - 3 = 0.14159265358979323846264... \quad x = 0.00000000000000000000000... \]

\[ \frac{1}{5} = 0.19999999999999999999999... \quad = 0.20000000000000000000000... \]

Representation is unique except for the cases that the decimal expansion ends in all 0’s or all 9’s. We will never use the all 9’s representation.
Proof that $[0,1)$ is not countable

Suppose, for the sake of contradiction, that there is a list of them:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>0.</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>...</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>...</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>...</td>
</tr>
<tr>
<td>$r_5$</td>
<td>0.</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
<tr>
<td>$r_6$</td>
<td>0.</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$r_7$</td>
<td>0.</td>
<td>7</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>8</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>...</td>
</tr>
<tr>
<td>$r_8$</td>
<td>0.</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td>4</td>
<td>...</td>
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<td>...</td>
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<td>...</td>
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<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Proof that \([0,1)\) is not countable

Suppose, for the sake of contradiction, that there is a list of them:

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ...
|---|---|---|---|---|---|---|---|---|---|---|
| \(r_1\) | 0. | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ...
| \(r_2\) | 0. | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | ...
| \(r_3\) | 0. | 1 | 4 | 2 | 8 | 5 | 7 | 1 | 4 | ...
| \(r_4\) | 0. | 1 | 4 | 1 | 5 | 9 | 2 | 6 | 5 | ...
| \(r_5\) | 0. | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | ...
| \(r_6\) | 0. | 2 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | ...
| \(r_7\) | 0. | 7 | 1 | 8 | 2 | 8 | 1 | 8 | 2 | ...
| \(r_8\) | 0. | 6 | 1 | 8 | 0 | 3 | 3 | 9 | 4 | ...
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ...
Proof that $[0,1)$ is not countable

Suppose, for the sake of contradiction, that there is a list of them:

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</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0.1</td>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.1</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$r_5$</td>
<td>0.1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
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<td>0.2</td>
<td>5</td>
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</tr>
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<td>1</td>
<td>8</td>
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</tr>
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<td>$r_8$</td>
<td>0.6</td>
<td>1</td>
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</tr>
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</table>

**Flipping rule:**

Only if the other driver deserves it.
Proof that \([0,1)\) is not countable

Suppose, for the sake of contradiction, that there is a list of them:

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<th>r_1</th>
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<tr>
<td>r_2</td>
<td>0.</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>r_3</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>r_4</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>r_5</td>
<td>0.</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
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<tr>
<td>r_6</td>
<td>0.</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>r_7</td>
<td>0.</td>
<td>7</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>r_8</td>
<td>0.</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>3</td>
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**Flipping rule:**
- If digit is 5, make it 1.
- If digit is not 5, make it 5.

...
Proof that \([0,1)\) is not countable

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</tr>
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<td>(r_6)</td>
<td>0.</td>
<td>2</td>
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<td>0</td>
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<tr>
<td>(r_7)</td>
<td>0.</td>
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<td>1</td>
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Flipping rule:
- If digit is 5, make it 1.
- If digit is not 5, make it 5.

If diagonal element is \(0.x_{11}x_{22}x_{33}x_{44}x_{55} \ldots\) then the flipped diagonal number call it \(d = 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots\) is also a real number in \([0,1)\).
Proof that $[0,1)$ is not countable

Suppose, for the sake of contradiction, that there is a list of them:

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<tr>
<td>$r_1$</td>
<td>0.</td>
<td>5</td>
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<td>0</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.</td>
<td>3</td>
<td>3</td>
<td>5</td>
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<td>$r_3$</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.</td>
<td>1</td>
<td>4</td>
<td>1</td>
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Flipping rule:
- If digit is 5, make it 1.
- If digit is not 5, make it 5.

For every $n \geq 1$:
- $r_n \neq d = 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots$
  - because the numbers differ on the $n$-th digit!

If diagonal element is $0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots$ then the flipped diagonal number call it $d = 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots$ is also a real number in $[0,1)$. 
**Proof that \([0,1)\) is not countable**

Suppose, for the sake of contradiction, that there is a list of them:

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Flipping rule:
- If digit is 5, make it 1.
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For every \(n \geq 1\):
\[
r_n \neq d = 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \ldots
\]
because the numbers differ on the \(n\)-th digit!

So the list is incomplete, which is a contradiction.

Thus the real numbers between 0 and 1 are **not countable**: “uncountable”
A note on this proof

• The set of rational numbers in $[0,1)$ also have decimal representations like this
  – The only difference is that rational numbers always have repeating decimals in their expansions $0.33333\ldots$ or $0.25000000\ldots$

• So why wouldn’t the same proof show that this set of rational numbers is uncountable?
A note on this proof

• The set of rational numbers in [0,1) also have decimal representations like this
  – The only difference is that rational numbers always have repeating decimals in their expansions 0.33333... or .25000000...

• So why wouldn’t the same proof show that this set of rational numbers is uncountable?
  – Given any listing (even one that is good like the dovetailing listing) we could create the flipped diagonal number $d$ as before
A note on this proof

• The set of rational numbers in [0,1) also have decimal representations like this
  – The only difference is that rational numbers always have repeating decimals in their expansions 0.33333... or .25000000...

• So why wouldn’t the same proof show that this set of rational numbers is uncountable?
  – Given any listing (even one that is good like the dovetailing listing) we could create the flipped diagonal number $d$ as before
  – However, $d$ would not have a repeating decimal expansion and so wouldn’t be a rational #
    It would not be a “missing” number, so no contradiction.
The set of all functions \( f : \mathbb{N} \to \{0, \ldots, 9\} \) is uncountable

\[
f(x) = x \mod 10.
\]
The set of all functions $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ is uncountable

Supposed listing of all the functions:

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ...
|---|---|---|---|---|---|---|---|---|---|---
| $f_1$ | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ...
| $f_2$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | ...
| $f_3$ | 1 | 4 | 2 | 8 | 5 | 7 | 1 | 4 |   | ...
| $f_4$ | 1 | 4 | 1 | 5 | 9 | 2 | 6 | 5 |   | ...
| $f_5$ | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 |   | ...
| $f_6$ | 2 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |   | ...
| $f_7$ | 7 | 1 | 8 | 2 | 8 | 1 | 8 | 2 |   | ...
| $f_8$ | 6 | 1 | 8 | 0 | 3 | 3 | 9 | 4 |   | ...
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
The set of all functions $f : \mathbb{N} \to \{0, \ldots, 9\}$ is uncountable

Supposed listing of all the functions:

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<tbody>
<tr>
<td>$f_1$</td>
<td>$5^1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_2$</td>
<td>3</td>
<td>$3^5$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1</td>
<td>4</td>
<td>$2^5$</td>
<td>8</td>
</tr>
<tr>
<td>$f_4$</td>
<td>1</td>
<td>4</td>
<td>1</td>
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<td>2</td>
<td>1</td>
<td>2</td>
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<tr>
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<td>2</td>
<td>5</td>
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<tr>
<td>$f_7$</td>
<td>7</td>
<td>1</td>
<td>8</td>
<td>2</td>
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<td>$f_8$</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

Flipping rule:
- If $f_n(n) = 5$, set $D(n) = 1$
- If $f_n(n) \neq 5$, set $D(n) = 5$
The set of all functions $f : \mathbb{N} \rightarrow \{0, ..., 9\}$ is uncountable

Supposed listing of all the functions:

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Flipping rule:
- If $f_n(n) = 5$, set $D(n) = 1$
- If $f_n(n) \neq 5$, set $D(n) = 5$

For all $n$, we have $D(n) \neq f_n(n)$. Therefore $D \neq f_n$ for any $n$ and the list is incomplete! $\Rightarrow \{f \mid f : \mathbb{N} \rightarrow \{0,1, ..., 9\}\}$ is not countable
We have seen that:

- The set of all (Java) programs is countable
- The set of all functions $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ is not countable

So: There must be some function $f : \mathbb{N} \rightarrow \{0, \ldots, 9\}$ that is not computable by any program!

Interesting... maybe.

Can we come up with an explicit function that is uncomputable?
Recall our language picture

All

Java

Context-Free

Binary Palindromes

Regular

DFA

NFA

Regex

Finite

0*

{001, 10, 12}

Java
Some Notation

We’re going to be talking about Java code.

\[ \text{CODE}(P) \] will mean “the code of the program \( P \)”

So, consider the following function:

```java
public String \( P \)(String \( x \)) {
    return new String(Arrays.sort(\( x \).toCharArray()));
}
```

What is \( P(\text{CODE}(P)) \)?

“((((()..;AACPSSaaabceegghiiilnnnnnooprrrrrrrrrrssstttttttuwwxxyy{)"
The Halting Problem

**Given:**  
- CODE(P) for any program P  
- input x

**Output:**  
- true if P halts on input x  
- false if P does not halt on input x

It turns out that it isn’t possible to write a program that solves the Halting Problem!
Proof by contradiction

• Suppose that $\textbf{H}$ is a Java program that solves the Halting problem. Then we can write this program:

```java
public static void $\textbf{D}(x)$ {
    if ($\textbf{H}(x,x) == \text{true}$) {
        while (true); /* don’t halt */
    }
    else {
        return; /* halt */
    }
}
```

• Does $\textbf{D}($\text{CODE}(\textbf{D})\text{)}$ halt?
Halting Problem

**Given:**
- \(\text{CODE}(P)\) for any program \(P\)
- input \(x\)

**Output:**
- \(\text{true}\) if \(P\) halts on input \(x\)
- \(\text{false}\) if \(P\) does not halt on input \(x\)

\(H\) solves the halting problem implies that
\(H(\text{CODE}(D),x)\) is \(\text{true}\) iff \(D(x)\) halts, \(H(\text{CODE}(D),x)\) is \(\text{false}\) iff not
Does $D(\text{CODE}(D))$ halt?

$H$ solves the halting problem implies that

$H(\text{CODE}(D), x)$ is $\text{true}$ iff $D(x)$ halts, $H(\text{CODE}(D), x)$ is $\text{false}$ iff not

```java
public static void D(x) {
    if ($H(x,x) == \text{true}$) {
        while (true); /* don’t halt */
    } else {
        return; /* halt */
    }
}
```

$H$ solves the halting problem implies that

$H(\text{CODE}(D), x)$ is $\text{true}$ iff $D(x)$ halts, $H(\text{CODE}(D), x)$ is $\text{false}$ iff not
H solves the halting problem implies that
\( H(\text{CODE}(D), x) \) is \textbf{true} iff \( D(x) \) halts,
\( H(\text{CODE}(D), x) \) is \textbf{false} iff not

Suppose that \( D(\text{CODE}(D)) \) \textbf{halts}.
Then, by definition of \( H \) it must be that
\[ H(\text{CODE}(D), \text{CODE}(D)) \] is \textbf{true}
Which by the definition of \( D \) means \( D(\text{CODE}(D)) \) \textbf{doesn’t halt}
Does $D(\text{CODE}(D))$ halt?

$H$ solves the halting problem implies that  
$H(\text{CODE}(D), x)$ is true iff $D(x)$ halts,  
$H(\text{CODE}(D), x)$ is false iff not $D(x)$ halts.

Suppose that $D(\text{CODE}(D))$ halts.  
Then, by definition of $H$ it must be that  
$H(\text{CODE}(D), \text{CODE}(D))$ is true  
Which by the definition of $D$ means $D(\text{CODE}(D))$ doesn’t halt.

Suppose that $D(\text{CODE}(D))$ doesn’t halt.  
Then, by definition of $H$ it must be that  
$H(\text{CODE}(D), \text{CODE}(D))$ is false  
Which by the definition of $D$ means $D(\text{CODE}(D))$ halts.

```java
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don’t halt */
    } else {
        return; /* halt */
    }
}
```

Does $D(\text{CODE}(D))$ halt?
H solves the halting problem implies that

\( H(\text{CODE}(D), x) \) is true iff \( D(x) \) halts, \( H(\text{CODE}(D), x) \) is false iff not

Suppose that \( D(\text{CODE}(D)) \) halts.

Then, by definition of \( H \) it must be that

\( H(\text{CODE}(D), \text{CODE}(D)) \) is true

Which by the definition of \( D \) means \( D(\text{CODE}(D)) \) doesn’t halt

Suppose that \( D(\text{CODE}(D)) \) doesn’t halt.

Then, by definition of \( H \) it must be that

\( H(\text{CODE}(D), \text{CODE}(D)) \) is false

Which by the definition of \( D \) means \( D(\text{CODE}(D)) \) halts

Contradiction!
We proved that there is no computer program that can solve the Halting Problem.

- There was nothing special about Java* [Church-Turing thesis]

This tells us that there is no compiler that can check our programs and guarantee to find any infinite loops they might have.
Connection to diagonalization

<table>
<thead>
<tr>
<th>All programs P</th>
<th>&lt;P₁&gt;</th>
<th>&lt;P₂&gt;</th>
<th>&lt;P₃&gt;</th>
<th>&lt;P₄&gt;</th>
<th>&lt;P₅&gt;</th>
<th>&lt;P₆&gt;</th>
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(P, x) entry is 1 if program P halts on input x and 0 if it runs forever.

Write <P> for CODE(P)
Connection to diagonalization

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(P,x) entry is 1 if program P halts on input x and 0 if it runs forever.

Behavior of program D would be like the flipped diagonal, so it can’t be in the list of all programs.

Contradiction!