Lecture 17: Recursively Defined Sets & Structural Induction
Recursive Definition of Sets

Recursive definition of set $S$

- **Basis Step:** $0 \in S$
- **Recursive Step:** If $x \in S$, then $x + 2 \in S$
- **Exclusion Rule:** Every element in $S$ follows from the basis step and a finite number of recursive steps.

We need the exclusion rule because otherwise $S = \mathbb{N}$ would satisfy the other two parts. However, we won’t always write it down on these slides.
Recursive Definitions of Sets

Basis: \( 6 \in S, 15 \in S \)

Recursive: If \( x, y \in S \), then \( x+y \in S \)

Basis: \([1, 1, 0] \in S, [0, 1, 1] \in S\)

Recursive: If \([x, y, z] \in S\), then \([\alpha x, \alpha y, \alpha z] \in S\) for any \( \alpha \in \mathbb{R} \)

If \([x_1, y_1, z_1] \in S\) and \([x_2, y_2, z_2] \in S\), then
\([x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S\).

Number of form \(3^n\) for \(n \geq 0\):
Recursive Definitions of Sets

Basis: $6 \in S, 15 \in S$

Recursive: If $x, y \in S$, then $x + y \in S$

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$

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If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Number of form $3^n$ for $n \geq 0$:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$. 
Recursive Definitions of Sets: General Form

Recursive definition

– **Basis step:** Some specific elements are in $S$
– **Recursive step:** Given some existing named elements in $S$ some new objects constructed from these named elements are also in $S$.
– **Exclusion rule:** Every element in $S$ follows from the basis step and a finite number of recursive steps
Strings

- An *alphabet* $\Sigma$ is any finite set of characters

- The set $\Sigma^*$ of *strings* over the alphabet $\Sigma$ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ ($\varepsilon$ is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$
Palindromes

Palindromes are strings that are the same backwards and forwards

**Basis:**

ε is a palindrome and any \( a \in \Sigma \) is a palindrome

**Recursive step:**

If \( p \) is a palindrome then \( apa \) is a palindrome for every \( a \in \Sigma \)
All Binary Strings with no 1’s before 0’s
All Binary Strings with no 1’s before 0’s

Basis:
\[ \varepsilon \in S \]

Recursive:
If \( x \in S \), then \( 0x \in S \)
If \( x \in S \), then \( x1 \in S \)
Functions on Recursively Defined Sets (on $\Sigma^*$)

**Length:**

$$\text{len}(\varepsilon) = 0$$

$$\text{len}(wa) = 1 + \text{len}(w) \text{ for } w \in \Sigma^*, \ a \in \Sigma$$

**Reversal:**

$$\varepsilon^R = \varepsilon$$

$$(wa)^R = aw^R \text{ for } w \in \Sigma^*, \ a \in \Sigma$$

**Concatenation:**

$$x \cdot \varepsilon = x \text{ for } x \in \Sigma^*$$

$$x \cdot wa = (x \cdot w)a \text{ for } x \in \Sigma^*, \ a \in \Sigma$$

**Number of c’s in a string:**

$$\#_c(\varepsilon) = 0$$

$$\#_c(wc) = \#_c(w) + 1 \text{ for } w \in \Sigma^*$$

$$\#_c(wa) = \#_c(w) \text{ for } w \in \Sigma^*, \ a \in \Sigma, \ a \neq c$$
Rooted Binary Trees

- **Basis:** is a rooted binary tree

- **Recursive step:**

If $T_1$ and $T_2$ are rooted binary trees, then $T$ also is a rooted binary tree.
Defining Functions on Rooted Binary Trees

- \( \text{size}(\cdot) = 1 \)

- \( \text{size} \left( \begin{array}{c} T_1 \\ T_2 \end{array} \right) = 1 + \text{size}(T_1) + \text{size}(T_2) \)

- \( \text{height}(\cdot) = 0 \)

- \( \text{height} \left( \begin{array}{c} T_1 \\ T_2 \end{array} \right) = 1 + \max\{\text{height}(T_1), \text{height}(T_2)\} \)
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the *Basis step*.

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of each of the existing named elements mentioned in the *Recursive step*.

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis.

**Conclude** that $\forall x \in S, P(x)$.
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all **specific** elements $u$ of $S$ mentioned in the **Basis step**

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of each of the **existing named elements** mentioned in the **Recursive step**

**Inductive Step:** Prove that $P(w)$ holds for each of the **new elements** $w$ constructed in the **Recursive step** using the **named elements** mentioned in the **Inductive Hypothesis**

**Conclude** that $\forall x \in S, P(x)$
Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

**Basis:** $0 \in \mathbb{N}$

**Recursive step:** If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most $n$ recursive steps, $P(x)$ is true.”
Using Structural Induction

- Let $S$ be given by...
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

**Claim:** Every element of $S$ is divisible by 3.
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 | x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive Step: Goal: Show $P(x + y)$.

   Since $P(x)$ is true, $3 \mid x$ and so $x = 3m$ for some integer $m$.

   Since $P(y)$ is true, $3 \mid y$ and so $y = 3n$ for some integer $n$.

   Therefore $x + y = 3m + 3n = 3(m + n)$ and thus $3 \mid (x + y)$.

   Hence $P(x + y)$ is true.

5. Therefore by induction $3 \mid x$ for all $x \in S$.

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Recursive: if $x, y \in S$ then $x + y \in S$. 
Claim: Every element of $S$ is divisible by 3.

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2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.
3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.
4. Inductive Step: **Goal:** Show $P(x+y)$.

\[ x=3m \quad \text{and} \quad y=3n \]

Therefore $x+y=3(m+n)$ and thus $3 \mid (x+y)$.

Hence $P(x+y)$ is true.

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Recursive: if $x, y \in S$ then $x + y \in S$.
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   Since $P(x)$ is true, $3 \mid x$ and so $x = 3m$ for some integer $m$ and
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   Therefore $x+y = 3m+3n = 3(m+n)$ and thus $3 \mid (x+y)$.

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Basis: $6 \in S; \ 15 \in S;

Recursive: if $x, y \in S$ then $x + y \in S$
Claim: $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

Let $P(y)$ be “$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$”.

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.
**Claim:** \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

**Base Case:** \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.
**Claim:** len(x•y) = len(x) + len(y) for all x, y ∈ Σ*

Let P(y) be “len(x•y) = len(x) + len(y) for all x ∈ Σ*”. We prove P(y) for all y ∈ Σ* by structural induction.

**Base Case:** y = ε. For any x ∈ Σ*, len(x•ε) = len(x) = len(x) + len(ε) since len(ε) = 0. Therefore P(ε) is true.

**Inductive Hypothesis:** Assume that P(w) is true for some arbitrary w ∈ Σ*

**Inductive Step:** Goal: Show that P(wa) is true for every a ∈ Σ
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

Base Case: \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.

Inductive Hypothesis: Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

Inductive Step: Goal: Show that \( P(wa) \) is true for every \( a \in \Sigma \).

Let \( a \in \Sigma \). Let \( x \in \Sigma^* \). Then \( \text{len}(x \cdot wa) = \text{len}((x \cdot w)a) \) by defn of \( \cdot \)

\[ = \text{len}(x \cdot w) + 1 \text{ by defn of len} \]
\[ = \text{len}(x) + \text{len}(w) + 1 \text{ by I.H.} \]
\[ = \text{len}(x) + \text{len}(wa) \text{ by defn of len} \]

Therefore \( \text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa) \) for all \( x \in \Sigma^* \), so \( P(wa) \) is true.

So, by induction \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \).
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet) = 1$, $\text{height}(\bullet) = 0$ and $1 = 2^1 - 1 = 2^{0+1} - 1$ so $P(\bullet)$ is true.
**Claim:** For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ and $1=2^{1}-1=2^{0+1}-1$ so $P(\bullet)$ is true.

3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees $T_1$ and $T_2$.

4. Inductive Step: Goal: Prove $P(\text{ } )$. 

By defn, $\text{size}(\ ) = 1 + \text{size}(T_1) + \text{size}(T_2) \\ \leq 1 + 2^{\text{height}(T_1)} + 1 - 1 + 2^{\text{height}(T_2)} + 1 - 1 \\ \leq 2^{\text{max(height}(T_1), \text{height}(T_2))} + 1 - 1 \\ \leq 2^{\text{height}(\ )} + 1 - 1 \\ \text{which is what we wanted to show.}$

5. So, the $P(T)$ is true for all rooted binary trees by structural induction.
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ and $1=2^{1}–1=2^{0+1}–1$ so $P(\bullet)$ is true.

3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees $T_1$ and $T_2$.

4. Inductive Step: Goal: Prove $P(\text{ } )$.

   By defn, $\text{size}(\text{ } ) = 1 + \text{size}(T_1) + \text{size}(T_2)$
   
   $\leq 1 + 2^{\text{height}(T_1)+1}–1 + 2^{\text{height}(T_2)+1}–1$

   by IH for $T_1$ and $T_2$

   $\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1}–1$

   $\leq 2(2^{\text{max}(\text{height}(T_1),\text{height}(T_2))+1})–1$

   $\leq 2(2^{\text{height}(\text{ } )})–1 \leq 2^{\text{height}(\text{ } )}+1 –1$

   which is what we wanted to show.

5. So, the $P(T)$ is true for all rooted bin. trees by structural induction.