Lecture 7: Logical Inference

Regrade Request:
Do it till Friday.

Wow, I can’t find fault with your proof.

You’ve shown the inconsistency—and thus invalidity—of basic logic itself.

Excellent, on to step two...

Dear Dr. Knuth,

I am writing to collect from you the $3,372,564.48 I am owed for discovering 1,317,408 errors in the art of computer programming...
Logical Inference

• So far we’ve considered:
  – How to understand and express things using propositional and predicate logic
  – How to compute using Boolean (propositional) logic
  – How to show that different ways of expressing or computing them are equivalent to each other

• Logic also has methods that let us infer implied properties from ones that we know
  – Equivalence is a small part of this
Applications of Logical Inference

• **Software Engineering**
  – Express desired properties of program as set of logical constraints
  – Use inference rules to show that program implies that those constraints are satisfied

• **Artificial Intelligence**
  – Automated reasoning

• **Algorithm design and analysis**
  – e.g., Correctness, Loop invariants.

• **Logic Programming, e.g. Prolog**
  – Express desired outcome as set of constraints
  – Automatically apply logic inference to derive solution
Proofs

• Start with hypotheses and facts
• Use rules of inference to extend set of facts
• Result is proved when it is included in the set
An inference rule: *Modus Ponens*

- If $A$ and $A \rightarrow B$ are both true then $B$ must be true.

- Write this rule as $A; A \rightarrow B \therefore B$.

- Given:
  - If it is Wednesday then you have a 311 class today.
  - It is Wednesday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.
Show that \( r \) follows from \( p, p \rightarrow q, \) and \( q \rightarrow r \)

1. \( p \)  Given
2. \( p \rightarrow q \)  Given
3. \( q \rightarrow r \)  Given
4. \( q \)  \text{MP of 1, 2.}
5. \( r \)  \text{MP of 3, 4}

Modus Ponens  \( A ; A \rightarrow B \)

\( \therefore B \)
Show that $r$ follows from $p$, $p \rightarrow q$, and $q \rightarrow r$

1. $p$ Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4. $q$ MP: 1, 2
5. $r$ MP: 3, 4
Show that \( \neg p \) follows from \( p \rightarrow q \) and \( \neg q \)

1. \( p \rightarrow q \)  
   Given
2. \( \neg q \)  
   Given
3. \( \neg q \rightarrow \neg p \)  
   Contrapositive: 1
4. \( \neg p \)  
   MP: 2, 3

Modus Ponens \( A ; A \rightarrow B \)  
\[ \therefore B \]
Inference Rules

If A is true and B is true ....

Requirements:  

Conclusions: 

Then, C must be true  

Then D must be true

Example (Modus Ponens): 

A ; A → B  

∴ B  

If I have A and A → B both true, 
Then B must be true.
Axioms: Special inference rules

If I have nothing...

Requirements:  

Conclusions: \( \therefore C, D \)

Then, \( C \) must be true
Then \( D \) must be true

Example (Excluded Middle):

\[ \therefore A \lor \neg A \]

\( A \lor \neg A \) must be true.
Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

\[ \begin{align*}
A \land B & \quad \text{Elim} \land \\
\therefore A, B & \\
\hline
\end{align*} \]

\[ \begin{align*}
A \lor B ; \neg A & \quad \text{Elim} \lor \\
\therefore B & \\
\hline
\end{align*} \]

\[ \begin{align*}
A ; A \rightarrow B & \quad \text{Modus Ponens} \\
\therefore B & \\
\hline
\end{align*} \]

\[ \begin{align*}
A ; B & \quad \text{Intro} \land \\
\therefore A \land B & \\
\hline
\end{align*} \]

\[ \begin{align*}
A & \quad \text{Intro} \lor \\
\therefore A \lor B, B \lor A & \\
\hline
\end{align*} \]

\[ \begin{align*}
A \Rightarrow B & \quad \text{Direct Proof Rule} \\
\therefore A \rightarrow B & \\
\hline
\end{align*} \]

Not like other rules
Proofs

Show that \( r \) follows from \( p, p \rightarrow q \) and \( (p \land q) \rightarrow r \)

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

1. \( p \) \hspace{1cm} \text{Given}
2. \( p \rightarrow q \) \hspace{1cm} \text{Given}
3. \( (p \land q) \rightarrow r \) \hspace{1cm} \text{Given}
4. \( q \) \hspace{1cm} \text{MP of 1,2}
5. \( p \land q \) \hspace{1cm} \text{Intro} \land 1,4
6. \( r \) \hspace{1cm} \text{MP of 3,5}

\[ A ; A \rightarrow B \]
\[ \therefore B \]

\[ A \land B \]
\[ \therefore A, B \]

\[ A ; B \]
\[ \therefore A \land B \]
Prove that $r$ follows from $p, p \rightarrow q$, and $p \land q \rightarrow r$.

1. $p$  
   Given

2. $p \rightarrow q$  
   Given

3. $q$  
   MP: 1, 2

4. $p \land q$  
   Intro $\land$: 1, 3

5. $p \land q \rightarrow r$  
   Given

6. $r$  
   MP: 4, 5

Two visuals of the same proof.
We will use the top one, but if the bottom one helps you think about it, that’s great!

$p; p \rightarrow q$

$p; q$

$p \land q$

$p \land q \rightarrow r$

$r$
Important: Applications of Inference Rules

• You can use *equivalences* to make substitutions of any sub-formula.

• Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. \( p \rightarrow r \) given

  2. \( (p \lor q) \rightarrow r \) intro \( \lor \) from 1.

  Does not follow! e.g. \( p=F, q=T, r=F \)
Prove that \( \neg r \) follows from \( p \land s, q \rightarrow \neg r, \) and \( \neg s \lor q. \)

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

Idea: Work backwards!

First: Write down givens and goal

20. \( \neg r \)
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

**Proofs**

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

**Idea: Work backwards!**

We want to eventually get \( \neg r \). How?

- We can use \( q \rightarrow \neg r \) to get there.
- The justification between 2 and 20 looks like “elim \( \rightarrow \)” which is MP.

20. \( \neg r \) MP: 2, ?
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

Idea: Work backwards!
We want to eventually get \( \neg r \). How?
- Now, we have a new “hole”
- We need to prove \( q \)...
  - Notice that at this point, if we prove \( q \), we’ve proven \( \neg r \)...

19. \( q \)  
20. \( \neg r \) MP: 2, 19
Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1. $p \land s$  Given
2. $q \rightarrow \neg r$  Given
3. $\neg s \lor q$  Given

This looks like or-elimination.

19. $q$  \[ ? \]
20. $\neg r$  MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

18. \( \neg \neg s \) \( ? \) \( \neg \neg s \) doesn’t show up in the givens but \( s \) does and we can use equivalences

19. \( q \) \lor Elim: 3, 18

20. \( \neg r \) MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \to \neg r \), and \( \neg s \lor q \).

1. \( p \land s \)  
   Given
2. \( q \to \neg r \)  
   Given
3. \( \neg s \lor q \)  
   Given

17. \( s \)  
18. \( \neg \neg s \)  
   Double Negation: 17
19. \( q \)  
   \( \lor \) Elim: 3, 18
20. \( \neg r \)  
   MP: 2, 19
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given

17. \( s \) \( \land \) Elim: 1
18. \( \neg \neg s \) Double Negation: 17
19. \( q \) \( \lor \) Elim: 3, 18
20. \( \neg r \) MP: 2, 19

No holes left! We just need to clean up a bit.
Prove that \( \neg r \) follows from \( p \land s \), \( q \rightarrow \neg r \), and \( \neg s \lor q \).

1. \( p \land s \) Given
2. \( q \rightarrow \neg r \) Given
3. \( \neg s \lor q \) Given
4. \( s \) \land Elim: 1
5. \( \neg \neg s \) Double Negation: 4
6. \( q \) \lor Elim: 3, 5
7. \( \neg r \) MP: 2, 6
To Prove An Implication: \( A \rightarrow B \)

- We use the direct proof rule
- The “pre-requisite” \( A \Rightarrow B \) for the direct proof rule is a proof that “Given \( A \), we can prove \( B \).”
- The direct proof rule:

  If you have such a proof then you can conclude that \( A \rightarrow B \) is true

Example: Prove \( p \rightarrow (p \lor q) \).

1. \( p \) Assumption
2. \( p \lor q \) Intro \( \lor \): 1
3. \( p \rightarrow (p \lor q) \) Direct Proof Rule
Proofs using the direct proof rule

Show that \( p \rightarrow r \) follows from \( q \) and \( (p \land q) \rightarrow r \)

1. \( q \) Given
2. \( (p \land q) \rightarrow r \) Given

\[
\begin{array}{ll}
3.1. & p \quad \text{Assumption} \\
3.2. & p \land q \quad \text{Intro } \land: 1, 3.1 \\
3.3. & r \quad \text{MP: 2, 3.2} \\
\end{array}
\]

If we know \( p \) is true...
Then, we’ve shown \( r \) is true

3. \( p \rightarrow r \) Direct Proof Rule

This is a proof of \( p \rightarrow r \)
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens...
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

1.1 \(p \land q\)  Assumption
1.2 \(p\)  elim \(\land\) in 1.1

1.10 \(p \lor q\)  Intro \(\lor\) in 1.2
1 \((p \land q) \rightarrow (p \lor q)\)  Direct Proof Rule.
Example

Prove: \((p \land q) \rightarrow (p \lor q)\)

1.1. \(p \land q\)  Assumption
1.2. \(p\)  Elim \(\land\): 1.1
1.3. \(p \lor q\)  Intro \(\lor\): 1.2

1. \((p \land q) \rightarrow (p \lor q)\)  Direct Proof Rule
Example

Prove: \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

1.1. \(((p \rightarrow q) \land (q \rightarrow r))\) \hspace{1cm} \text{Assumption}
1.2. \(p \rightarrow q\) \hspace{1cm} \text{elim of } \land \hspace{1cm} 1.1
1.3. \(q \rightarrow r\) \hspace{1cm} \text{elim of } \land \hspace{1cm} 1.1

1.20.1. \(p\) \hspace{1cm} \text{Assumption}
1.20.2. \(q\) \hspace{1cm} \text{MP of } 1.2, 1.20.1

1.20.20. \(r\) \hspace{1cm} \text{MP of } 1.3 \text{ and } 1.20.2

1.20. \(p \rightarrow r\) \hspace{1cm} \text{DPR}

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\) \hspace{1cm} \text{DPR}
Example

Prove: \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

1.1. \((p \rightarrow q) \land (q \rightarrow r)\)  Assumption
1.2. \(p \rightarrow q\)  \land\ Elim: 1.1
1.3. \(q \rightarrow r\)  \land\ Elim: 1.1

1.4.1. \(p\)  Assumption
1.4.2. \(q\)  MP: 1.2, 1.4.1
1.4.3. \(r\)  MP: 1.3, 1.4.2

1.4. \(p \rightarrow r\)  Direct Proof Rule

1. \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)  Direct Proof Rule
One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.