Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:
\[ 0 \cup ((1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*) \]

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:
\[ 0 \cup ((1 \cup 2)(0 \cup 1 \cup 2)^*0) \]

(c) Write a regular expression that matches all binary strings that contain the substring ”111”, but not the substring ”000”.

Solution:
\[ (01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon) \]

CFGs

Construct CFGs for the following languages:

(a) All binary strings that end in 00.

Solution:
\[ S \rightarrow 0S \mid 1S \mid 00 \]

(b) All binary strings that contain at least three 1’s.

Solution:
\[ S \rightarrow TTT \\
T \rightarrow 0T \mid T0 \mid 1T \mid 1 \]

(c) All binary strings with an equal number of 1’s and 0’s.

Solution:
\[ S \rightarrow 0S1S \mid 1S0S \mid \varepsilon \\
S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon \]
Relations

(a) Draw the transitive-reflexive closure of \{(1, 2), (2, 3), (3, 4)\}.

\[ \begin{array}{cccc}
1 & 
\rightarrow &
2 &
\rightarrow &
3 &
\rightarrow &
4
\end{array} \]

Solution:

(b) Suppose that \( R \) is reflexive. Prove that \( R \subseteq R^2 \).

Solution: Suppose \((a, b) \in R\). Since \( R \) is reflexive, we know \((b, b) \in R\) as well. Since there is a \( b \) such that \((a, b) \in R \) and \((b, b) \in R\), it follows that \((a, b) \in R^2\). Thus, \( R \subseteq R^2 \).

(c) Consider the relation \( R = \{(x, y) : x = y + 1\} \) on \( \mathbb{N} \). Is \( R \) reflexive? Transitive? Symmetric? Anti-symmetric?

Solution: It isn’t reflexive, because \( 1 \neq 1 + 1 \); so, \((1, 1) \notin R\). It isn’t symmetric, because \((2, 1) \in R\) (because \( 2 = 1 + 1 \)), but \((1, 2) \notin R\), because \( 1 \neq 2 + 1 \). It isn’t transitive, because note that \((3, 2) \in R \) and \((2, 1) \in R \), but \((3, 1) \notin R\). It is anti-symmetric, because consider \((x, y) \in R\) such that \( x \neq y \). Then, \( x = y + 1 \) by definition of \( R \). However, \((y, x) \notin R\), because \( y = x - 1 \neq x + 1 \).

(d) Consider the relation \( S = \{(x, y) : x^2 = y^2\} \) on \( \mathbb{R} \). Prove that \( S \) is reflexive, transitive, and symmetric.

Solution: Consider \( x \in \mathbb{R} \). Note that by definition of equality, \( x^2 = x^2 \); so, \((x, x) \in R\); so, \( R \) is reflexive.

Consider \((x, y) \in R\). Then, \( x^2 = y^2 \). It follows that \( y^2 = x^2 \); so, \((y, x) \in R\). So, \( R \) is symmetric.

Suppose \((x, y) \in R \) and \((y, z) \in R\). Then, \( x^2 = y^2 \), and \( y^2 = z^2 \). Since equality is transitive, \( x^2 = z^2 \). So, \((x, z) \in R\). So, \( R \) is transitive.