1. Strong Induction

(a) Prove that, for all \( n \in \mathbb{N} \), every \( n \) has an unsigned binary representation.

**Solution:**
Let \( P(n) \) be “\( n \) has an unsigned binary representation”. We will prove \( P(n) \) for all integers \( n \in \mathbb{N} \) by induction.

**Base Case** \((n = 0)\): The unsigned binary representation of 0 is \( 0_2 \), so \( P(0) \) holds.

**Induction Hypothesis:** Assume that \( P(j) \) holds for all integers \( 0 \leq j \leq k \) for some arbitrary \( k \in \mathbb{N} \).

**Induction Step:** Goal: Show \( P(k + 1) \), i.e., \( k + 1 \) has an unsigned binary representation

Let \( 2^\ell \) be the largest power of two not greater than \( k + 1 \) (i.e. \( \ell = \lceil \log_2(n) \rceil \)). Let \( r = k + 1 - 2^\ell \), the remainder.

Note that \( r < 2^\ell \), so \( r \) has some binary representation \( r_2 \) [by the Induction Hypothesis].

Then \( 1r_2 \) is the binary expansion for \( k + 1 \). This proves \( P(k + 1) \).

**Conclusion:** \( P(n) \) holds for all integers \( n \in \mathbb{N} \) by induction.

(b) Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function \( f \):

\[
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 1 \\
  f(n) &= 2f(n - 1) - f(n - 2) \quad \text{for } n \geq 2
\end{align*}
\]

Determine, with proof, the number, \( f(n) \), of rabbits that Cantelli owns in year \( n \).

**Solution:**
Let \( P(n) \) be “\( f(n) = n \)”. We prove that \( P(n) \) is true for all \( n \in \mathbb{N} \) by strong induction on \( n \).

**Base Case** \((n = 0)\): \( f(0) = 0 \) by definition. So, \( P(0) \) holds.

**Induction Hypothesis:** Assume that for some arbitrary integer \( k \geq 0 \), \( P(j) \) holds for all \( 0 \leq j \leq k \).

**Induction Step:** We show \( P(k + 1) \).

Case 1 \((k = 0)\): Then, by definition \( f(k + 1) = f(1) = 1 \). So, \( P(k + 1) \) holds.

Case 2\((k \geq 1)\): Since \( k + 1 \geq 2 \), by definition of \( f \),

\[
  f(k + 1) = 2f(k) - f(k - 1)
\]

Since \( 0 \leq k - 1, k \leq k \), by induction hypothesis,

\[
  f(k + 1) = 2(k) - (k - 1) = k + 1
\]

This proves \( P(k + 1) \).

Therefore, \( P(n) \) is true for all \( n \in \mathbb{N} \).
2. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: "" is a string

Recursive Step: If $X$ is a string and $c$ is a character then $\text{append}(c, X)$ is a string.

Recall the following recursive definition of the function $\text{len}$:

\[
\begin{align*}
\text{len}("") &= 0 \\
\text{len}(\text{append}(c, X)) &= 1 + \text{len}(X)
\end{align*}
\]

Now, consider the following recursive definition:

\[
\begin{align*}
\text{double}("") &= "" \\
\text{double}(\text{append}(c, X)) &= \text{append}(c, \text{append}(c, \text{double}(X)))
\end{align*}
\]

Prove that for any string $X$, $\text{len}(\text{double}(X)) = 2\text{len}(X)$.

Solution:

For a string $X$, let $P(X)$ be "\text{len}(\text{double}(X)) = 2\text{len}(X)". We prove $P(X)$ for all strings $X$ by structural induction.

Base Case. We show $P(\"\")$ holds. By definition $\text{len}(\text{double}(\"\")) = \text{len}(\"\") = 0$. On the other hand, $2\text{len}(\"\") = 0$ as desired.

Induction Hypothesis. Suppose $P(X)$ holds for some string $X$.

Induction Step. We show that $P(\text{append}(c, X))$ holds for any character $c$.

\[
\begin{align*}
\text{len}(\text{double}(\text{append}(c, X))) &= \text{len}(\text{append}(c, \text{append}(c, \text{double}(X)))) \\
&= 1 + \text{len}(\text{append}(c, \text{double}(X))) \\
&= 1 + 1 + \text{len}(\text{double}(X)) \\
&= 2 + 2\text{len}(X) \\
&= 2(1 + \text{len}(X)) \\
&= 2(\text{len}(\text{append}(c, X))) \\
\end{align*}
\]

This proves $P(\text{append}(c, X))$.

Thus, $P(X)$ holds for all strings $X$ by structural induction.

(b) Consider the following definition of a (binary) Tree:

Basis Step: $\bullet$ is a Tree.

Recursive Step: If $L$ is a Tree and $R$ is a Tree then $\text{Tree}(\bullet, L, R)$ is a Tree.

The function leaves returns the number of leaves of a Tree. It is defined as follows:

\[
\begin{align*}
\text{leaves}(\bullet) &= 1 \\
\text{leaves}(\text{Tree}(\bullet, L, R)) &= \text{leaves}(L) + \text{leaves}(R)
\end{align*}
\]

Also, recall the definition of size on trees:

\[
\begin{align*}
\text{size}(\bullet) &= 1 \\
\text{size}(\text{Tree}(\bullet, L, R)) &= 1 + \text{size}(L) + \text{size}(R)
\end{align*}
\]

Prove that $\text{leaves}(T) \geq \text{size}(T)/2$ for all Trees $T$. 
**Solution:**

In this problem, we define a strengthened predicate. For a tree \( T \), let \( P \) be \( \text{leaves}(T) \geq \text{size}(T)/2 + 1/2 \). We prove \( P \) for all trees \( T \) by structural induction.

**Base Case.** We show that \( P(\cdot) \) holds. By definition of \( \text{leaves}(\cdot) \), \( \text{leaves}(\bullet) = 1 \) and \( \text{size}(\bullet) = 1 \). So, \( \text{leaves}(\bullet) = 1 \geq 1/2 + 1/2 = \text{size}(\bullet)/2 + 1/2 \).

**Induction Hypothesis:** Suppose \( P(L) \) and \( P(R) \) hold for trees \( L, R \).

**Induction Step:** We prove \( P(\text{Tree}(\bullet, L, R)) \) holds.

\[
\begin{align*}
\text{leaves}(\text{Tree}(\bullet, L, R)) &= \text{leaves}(L) + \text{leaves}(R) & [\text{By Definition of leaves}] \\
&\geq \left(\text{size}(L)/2 + 1/2\right) + \left(\text{size}(R)/2 + 1/2\right) & [\text{By IH}] \\
&= 1 + \text{size}(\text{Tree}(\bullet, L, R))/2 & [\text{By Definition of size}] \\
&\geq \text{size}(\text{Tree}(\bullet, L, R))/2 + 1/2
\end{align*}
\]

This proves \( P(\text{Tree}(\bullet, X, R)) \).

Thus, the \( P(T) \) holds for all trees \( T \).