Spring 2015
Lecture 16: Strong induction
Evan covering office hours today [CSE 624, 2:30-3:30pm]

MIDTERM FRIDAY (IN THIS ROOM, USUAL TIME)

Closed book.
One page (front and back) of hand-written notes allowed.
Exam includes induction and strong induction!
Homework #5 is up now, but due on Friday, May 15th.

Review sessions:

James Wednesday @ 6PM [probably in EEB 105]
Additional: ...?
**review:** induction is a rule of inference

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

**Domain:** Natural Numbers
1. Prove $P(0)$
2. Let $k$ be an arbitrary integer $\geq 0$
   3. Assume that $P(k)$ is true
   4. ...
   5. Prove $P(k+1)$ is true

6. $P(k) \rightarrow P(k+1)$ \hspace{1cm} Direct Proof Rule
7. $\forall k \ (P(k) \rightarrow P(k+1))$ \hspace{1cm} Intro $\forall$ from 2-6
8. $\forall n \ P(n)$ \hspace{1cm} Induction Rule 1&7
review: format of an induction proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]

\[ \therefore \forall n \ P(n) \]

1. Prove P(0)  
2. Let k be an arbitrary integer \( \geq 0 \)
3. Assume that P(k) is true
4. ...
5. Prove P(k+1) is true
6. \( P(k) \rightarrow P(k+1) \)  
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \)  
8. \( \forall n \ P(n) \)
Proof:

1. “We will show that \( P(n) \) is true for every \( n \geq 0 \) by induction.”
2. “Base Case:” Prove \( P(0) \)
3. “Inductive Hypothesis:”
   
   Assume \( P(k) \) is true for some arbitrary integer \( k \geq 0 \)
4. “Inductive Step:” Want to prove that \( P(k+1) \) is true:
   
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it.
   (Don’t assume \( P(k+1) \) !)
5. “Conclusion: Result follows by induction.”
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

Place piece in the center and recurse on each quadrant!
Inductive proof:

\[ P(n) = "\text{A } 2^n \times 2^n \text{ checkerboard with one piece removed can be tiled by } \square \text{ pieces.} \]

Base case (Zen): \( n = 0 \). An empty board can be tiled with no pieces \( \Rightarrow P(0) \)

Alternate base case: \( n = 1 \). A \( 2 \times 2 \) board with one square missing can be tiled with one piece \( \Rightarrow P(1) \)

Inductive hypothesis: Assume \( P(k) \) for some \( k \geq 0 \).

Inductive step: Consider any \( 2^{k+1} \times 2^{k+1} \) board with one square missing. There exists a way to place a piece in the center so that each quadrant is a \( 2^k \times 2^k \) board with one square missing. By IH, there is a way to tile each of those four boards. Thus we can tile the \( 2^{k+1} \times 2^{k+1} \) board as well. We conclude that \( P(k + 1) \) holds.

By induction, \( P(n) \) holds for every \( n \geq 1 \) (or \( n \geq 0 \) if we started there).
Let $P(n)$ be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on $n$.

**Base Case:**

$3^3 = 27 \geq 9 = 3^2$. So, $P(3)$ is true.

**Induction Hypothesis:**

Suppose $P(k)$ is true for some arbitrary $k \geq 3$.

**Induction Step:**

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$

Since this is exactly $P(k+1)$, we’ve shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.
prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let $P(n)$ be “$2n^3 + 2n - 5 \geq n^2$” for all $n \geq 2$.

We go by induction on $n$.

**Base Case:**

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, $P(0)$ is true.

**Induction Hypothesis:**

Suppose $P(n)$ is true for some arbitrary $n \geq 2$.

**Induction Step:** Then, note that...

\[
(n+1)^2 \leq n^2 + 2n + 1 \\
\leq (2n^3 + 2n - 5) + 2n + 1 \quad \text{(by IH)} \\
\leq (2n^3 + 4n + 1) - 5 \quad \text{(Re-arranging)} \\
\leq (2n^3 + 6n^2 + 6n + 2) - 5 \quad \text{(4n + 1 \leq 6n + 6n^2 + 2)} \\
\leq 2(n+1)^3 - 5 \quad \text{(Factoring)} \\
\leq 2(n+1)^3 + 2n - 5 \quad \text{(0 \leq 2n)}
\]

Since this is exactly $P(k+1)$, we’ve shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.
strong induction

\[ P(0) \]
\[ \forall k \left( \left( P(0) \land P(1) \land P(2) \land \cdots \land P(k) \right) \rightarrow P(k + 1) \right) \]
\[ \therefore \forall n P(n) \]

Follows from ordinary induction applied to
\[ Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n) \]
1. By induction we will show that $P(n)$ is true for every $n \geq 0$

2. Base Case: Prove $P(0)$

3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every $j$ from 0 to $k$

4. Inductive Step:
   Prove that $P(k + 1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)

5. Conclusion: Result follows by induction
every integer at least 2 is the product of primes

We argue by strong induction.

\[ P(n) = \text{“n can be expressed as a product of primes”} \text{ for } n \geq 2. \]

**Base Case:**

Note that 2 is prime; so, we can express it as “2” which is a product of primes.

**Induction Hypothesis:**

Suppose \( P(2) \land P(3) \land \cdots \land P(k) \) is true for some \( k \geq 2 \).

**Induction Step:**

We go by cases.

Suppose \( k+1 \) is prime. Then, “\( k+1 \)” is a product of primes.

Suppose \( k+1 \) is composite. Then, \( k+1 = ab \) for some \( a \) and \( b \) such that \( 1 < a, b < k+1 \).

By our IH, we know \( a = p_1 p_2 \cdots p_m \) and \( b = q_1 q_2 \cdots q_n \).

So, \( k+1 = ab = \text{“}p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n\text{”} \), which is a product of primes.

Thus, our claim is true for \( n \geq 2 \) by strong induction.
recursive definition of functions

• \( F(0) = 0; \ F(n + 1) = F(n) + 1 \) for all \( n \geq 0 \)

• \( G(0) = 1; \ G(n + 1) = 2 \times G(n) \) for all \( n \geq 0 \)

• \( 0! = 1; \ (n + 1)! = (n + 1) \times n! \) for all \( n \geq 0 \)

• \( H(0) = 1; \ H(n + 1) = 2^{H(n)} \) for all \( n \geq 0 \)
Fibonacci numbers

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2 \]
Theorem: \( f_n < 2^n \) for all \( n \geq 2 \).

\[ P(n) = "f_n < 2^n" \]

Base case: \( f_2 = f_1 + f_0 = 1 < 4 = 2^2 \Rightarrow P(2) \)

(Strong) Induction hypothesis: Assume \( P(2), P(3), \ldots, P(n) \)

Inductive step: \( f_{n+1} = f_n + f_{n-1} \)

\[ \leq 2^n + 2^{n-1} \quad (\text{IH}) \]

\[ < 2 \cdot 2^n = 2^{n+1} \]

So by induction, \( P(n) \quad \forall n \geq 2 \).