Fall 2015
Lecture 15: Induction
prove: for all $n \geq 0$, $a$ is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.

Suppose that $a$ is odd. We know that if $a, b$ are odd, then $ab$ is also odd.

So: $\ldots ((a \cdot a) \cdot a) \ldots \cdot a = a^n$ $[n$ times$]$

Those “…”s are a problem! We’re trying to say “we can use the same argument over and over…”

We’ll come back to this.
mathematical induction

Method for proving statements about all integers $\geq 0$

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to use the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```java
for(int i=0; i < n; n++) { ... }
```
  - Show $P(i)$ holds after $i$ times through the loop

```java
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x-1)+1; }
}
```
  - $f(x) = x$ for all values of $x \geq 0$ naturally shown by induction.
induction is a rule of inference

Domain: Natural Numbers

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]

\[ \therefore \forall n \ P(n) \]
using the induction rule in a formal proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \)
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
   3. Assume that \( P(k) \) is true
   4. ...
   5. Prove \( P(k+1) \) is true
6. \( P(k) \rightarrow P(k+1) \) \hspace{1cm} Direct Proof Rule
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \) \hspace{1cm} \text{Intro} \ \forall \text{ from 2-6}
8. \( \forall n \ P(n) \) \hspace{1cm} \text{Induction Rule 1&7}
format of an induction proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]
\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \)
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
3. Assume that \( P(k) \) is true
4. ...
5. Prove \( P(k+1) \) is true
6. \( P(k) \rightarrow P(k+1) \)
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \)
8. \( \forall n \ P(n) \)
Can we describe the pattern?

\[ 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1 \]
proving $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

• We could try proving it normally...
  – We want to show that $1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$.
  – So, what do we do now? We can sort of explain the pattern, but that’s not a proof...
• We could prove it for $n=1$, $n=2$, $n=3$, ...
  (individually), but that would literally take forever...
Proof:

1. “We will show that \( P(n) \) is true for every \( n \geq 0 \) by induction.”
2. “Base Case:” Prove \( P(0) \)
3. “Inductive Hypothesis:”
   
   Assume \( P(k) \) is true for some arbitrary integer \( k \geq 0 \)

4. “Inductive Step:” Want to prove that \( P(k+1) \) is true:
   
   Use the goal to figure out what you need.
   
   Make sure you are using I.H. and point out where you are using it. (Don’t assume \( P(k+1) \) !)

5. “Conclusion: Result follows by induction.”
proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

**P(n)** = "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$"

**Goal:** ∀ integers $n \geq 0$, $P(n)$

**Base case:** $P(0)$ = "1 = $2^1 - 1$" true b/c $2-1 = 1$.

**IH:** Assume $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ $P(k)$ for some integer $k \geq 0$.

**IS:** From $P(k)$, we know

\[1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1.\]

Hence $P(k+1)$ holds.

**Conclusion:** Therefore by induction ∀ n $P(n)$. 
proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be “$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case $(n=0)$: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary $k \geq 0$.

4. Induction Step:

   **Goal:** Show $P(k+1)$, i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   \[
   1 + 2 + \ldots + 2^k = 2^{k+1} - 1 \text{ by IH}
   \]

   Adding $2^{k+1}$ to both sides, we get:

   \[
   1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1
   \]

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

5. Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.
another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- ...

$2^{2n} - 1$ is a multiple of 3 for all $n \geq 0$

$2^{2n} - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + 1)$
prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

\[ P(n) = "3 \mid 2^{2n} - 1" \]

Pf by induction that \( \forall n \geq 0 \ P(n) \)

Base case: \( P(0): 3 \mid \frac{2^0 - 1}{0} \)
\[ 0 = 3 \cdot 0. \]

IH: Assume that $3 \mid 2^{2k} - 1$ for some
\[ \text{Int. } k \geq 0 \]

IS: From (IH), $2^{2k} - 1 = 3j$ for some \[ \text{Int. } j \geq 0 \]

Thus \[ 2^2(2^{2k} - 1) = 2^2 \cdot 3j \]
\[ \Rightarrow \ 2^{2k+1} - 4 = 12j \]
\[ \Rightarrow \ 2^{2(k+1)} - 4 = 12j \]
\[ \Rightarrow \ 2^{2(k+1)} - 1 = 12j + 3 = 3(4j + 1) \]
(Corollary: $3j$ and $\text{in P(n)}$)
For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

\[ P(n) = \text{"A } 2^n \times 2^n \text{ checkerboard with one square removed can be tiled by\".}" \]

Base case: $P(0)$ We just have an empty $1 \times 1$ board which is tiled without doing anything.

IH: Assume $P(n)$ holds for some integer $n \geq 2$.

IS: Consider a $2^n \times 2^n + 1$ board with one square removed. By symmetry, we can assume:

Now place a block at the intersection

We now have four $2^n \times 2^n$ squares $S_1, S_2, S_3, S_4$ each with one block removed. By IH, each of $S_1, S_2, S_3, S_4$ can be tiled.

Thus the $2^n \times 2^n + 1$ square can be tiled.

\[ \Rightarrow P(n+1) \]
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

Assume it holds for $2^n \times 2^n$.
prove: \( n^n \geq n! \) for all \( n \geq 1 \)

\[
P(n) = "n^n \geq n!" \quad \text{with} \quad n! = n(n-1) \cdots 3 \cdot 2 \cdot 1
\]

**Base case:** \( P(1) \) is true:

\[1 = 1^1 \geq 1! = 1\]

**IH:** Assume that \( k^k \geq k! \) for some \( k \geq 1 \).

**IS:** \( k^k \geq k! \Rightarrow (k+1)^{k+1} \geq (k+1) k! = (k+1)! \)

\[(k+1)^{k+1} = (k+1) \cdot (k+1)^k \geq (k+1) k^k\]

Combine by prev. line \((k+1)^{k+1} \geq k!\)

**Conclusion:** By induction \( \forall n \in P(n) \).