prove: for all $n > 0$, $a$ is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.
Suppose that $a$ is odd. We know that if $a, b$ are odd, then $ab$ is also odd.

So: $\left( \cdots \left( (a \cdot a) \cdot a \right) \cdots \cdot a \right) = a^n \quad [n \text{ times}]$

Those “…”s are a problem! We’re trying to say “we can use the same argument over and over…”
We’ll come back to this.
mathematical induction

Method for proving statements about all integers $\geq 0$

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to use the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```java
for(int i=0; i < n; n++) { … }
```
- Show $P(i)$ holds after $i$ times through the loop

```java
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x-1)+1; }
}
```
- $f(x) = x$ for all values of $x \geq 0$ naturally shown by induction.
induction is a rule of inference

\( P(0) \)
\[ \forall k (P(k) \rightarrow P(k+1)) \]
\[ \therefore \forall n P(n) \]

\[ P(k) \overset{\text{def}}{=} Q(2k) \]

\[ P(0), \forall k P(k) \rightarrow P(k+1) \]
\[ \forall n \ Q(2n) \]
using the induction rule in a formal proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]

\[ \therefore \ \forall n \ P(n) \]

1. Prove \( P(0) \)
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
   3. Assume that \( P(k) \) is true
   4. ...
   5. Prove \( P(k+1) \) is true
6. \( P(k) \rightarrow P(k+1) \) \hspace{1cm} Direct Proof Rule
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \) \hspace{1cm} Intro \( \forall \) from 2-6
8. \( \forall n \ P(n) \) \hspace{1cm} Induction Rule 1&7
format of an induction proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]

\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \) **Base Case**

2. Let \( k \) be an arbitrary integer \( \geq 0 \)
3. Assume that \( P(k) \) is true **Inductive Hypothesis**
4. ...
5. Prove \( P(k+1) \) is true **Inductive Step**

6. \( P(k) \rightarrow P(k+1) \) Direct Proof Rule
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \) Intro \( \forall \) from 2-6
8. \( \forall n \ P(n) \) Induction Rule 1&7

Conclusion
1 + 2 + 4 + 8 + \cdots + 2^n

- 1 = 1
- 1 + 2 = 3
- 1 + 2 + 4 = 7
- 1 + 2 + 4 + 8 = 15
- 1 + 2 + 4 + 8 + 16 = 31

Can we describe the pattern?

\[ 1 + 2 + \cdots + 2^n = 2^{n+1} - 1 \]
We could try proving it normally…
   – We want to show that $1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$.
   – So, what do we do now? We can sort of explain the pattern, but that’s not a proof…

We could prove it for $n=1, n=2, n=3, \ldots$ (individually), but that would literally take forever…
Inductive proof in five easy steps

Proof:

1. “We will show that P(n) is true for every n ≥ 0 by induction.”
2. “Base Case:” Prove P(0)
3. “Inductive Hypothesis:”
   Assume P(k) is true for some arbitrary integer k ≥ 0”
4. “Inductive Step:” Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it.
   (Don’t assume P(k+1) !)
5. “Conclusion: Result follows by induction.”
proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

$P(n) = \{1 + 2 + \ldots + 2^n = 2^{n+1} - 1\}$

Goal: $\forall n P(n)$  
Domain: $\text{nat, numbers}$

Base case: $P(0)$ is "$1 = 2^1 - 1 = 1"$ true.

Inductive hypothesis: Assume $P(k)$ for some $k \geq 0$

Inductive step: By $P(k)$, we know

$$1 + 2 + \ldots + 2^k = 2^{k+1} - 1$$

$$\Rightarrow 1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$\Rightarrow P(k+1)$.

By induction $\forall k P(k)$. 

1. Let $P(n)$ be “$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.

2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$

3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary $k \geq 0$.

4. Induction Step:
   
   **Goal:** Show $P(k+1)$, i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   Adding $2^{k+1}$ to both sides, we get:
   
   
   
   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

5. Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.
another example of a pattern

- \(2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0\)
- \(2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1\)
- \(2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5\)
- \(2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21\)
- \(2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85\)
- ...

\[
\begin{align*}
3 & \mid 2^k - 1
\end{align*}
\]
Prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

**Base case:** $P(0) = "3 \mid 2^0 - 1"

$\equiv "3 \mid 0"$ true b/c $0 = 0 \cdot 3$

**I H:** Assume that $3 \mid 2^{2k} - 1$ for some $k \geq 0$

**I S:** $2^{2k} - 1 = 3a$ for some integer $a$ by I H.

$\Rightarrow 4(2^{2k} - 1) = 12a$

$\Rightarrow 2^2(2^{2k} - 1) = 2^{2(k+1)} - 4 = 12a$

$\Rightarrow 2^{2(k+1)} - 1 = 12a + 3 = 3(4a + 1)$
For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

Base case: $P(0)$

$P(n) = \text{tiles } 2^n \times 2^n \text{ after one square missing}$

$2^{n+1} \times 2^{n+1}$
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

\[ P(n) = "A 2^n \times 2^n \text{ checkerboard with one square removed can be tiled by } \square."

Base case: $P(0)$  We just have an empty 1x1 board which is tiled without doing anything.

IH: Assume $P(n)$ holds for some integer $n \geq 0$.

IS: Consider a $2^{n+1} \times 2^{n+1}$ board with one square removed.

By symmetry, we can assume:

Now place a block at the intersection

We now have four $2^n \times 2^n$ squares $S_1, S_2, S_3, S_4$

each with one block removed. By IH, each of $S_1, S_2, S_3, S_4$ can be tiled.

Thus the $2^{n+1} \times 2^{n+1}$ square can be tiled.

\[ \Rightarrow P(n+1) \]
prove: $n^n \geq n!$ for all $n \geq 1$