Instructions:

• Closed book, closed notes, no cell phones, no calculators.

• You have **110 minutes** to complete the exam.

• Answer all problems on the exam paper.

• If you need extra space use the back of a page.

• Problems are not of equal difficulty; if you get stuck on a problem, move on.

• You may tear off the last two pages of equivalence and inference rules. These must be handed in at the end but will not be graded.

• It may be to your advantage to read all the problems before beginning the exam.
1. [? points]
Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

Solution: Let $D$ be an arbitrary DFA. Consider $S = \{0^n : n \geq 0\}$. Since $S$ is infinite and $D$ has finitely many states, we know $0^i \in S$ and $0^j \in S$ both end in the same state for some $i < j$. Append $1^j$ to both strings to get:

$a = 0^i 1^j$ Note that $a \in L$, because $i < j$ and $0^i 1^j \in \Sigma^*$.

$b = 0^j 1^j$ Note that $b \notin L$, because $j \notin L$.

Since $a$ and $b$ both end in the same state, and that state cannot both be an accept and reject state, $D$ cannot recognize $L$. Since $D$ was arbitrary, no DFA recognizes $L$; so, $L$ is irregular.
2. [? points]
Define
\[
T(n) = \begin{cases} 
  n & \text{if } n = 0, 1 \\
  4T(\lfloor \frac{n}{2} \rfloor) + n & \text{otherwise}
\end{cases}
\]

Prove that \(T(n) \leq n^3\) for \(n \geq 3\).

Solution: We go by strong induction on \(n\). Let \(P(n)\) be "\(T(n) \leq n^3\)" for \(n \in \mathbb{N}\).

Base Cases. When \(n = 3\), \(T(3) = 4T(\lfloor \frac{3}{2} \rfloor) + 3 = 4T(1) + 3 = 7 \leq 27 = 3^3\).
When \(n = 4\), \(T(4) = 4T(\lfloor \frac{4}{2} \rfloor) + 4 = 4T(2) + 4 = 27 \leq 64 = 4^3\).
When \(n = 5\), \(T(5) = 4T(\lfloor \frac{5}{2} \rfloor) + 5 = 4T(2) + 5 = 28 \leq 4^4\).

Induction Hypothesis. Suppose \(P(3) \land P(4) \land \cdots \land P(k)\) for some \(k \geq 5\).

Induction Step. We want to prove \(P(k + 1)\): Note that
\[
T(k + 1) = 4T(\lfloor \frac{k + 1}{2} \rfloor) + k + 1, \quad \text{because } k + 1 \geq 2.
\]
\[
\leq 4 \left( \frac{k + 1}{2} \right)^3 + k + 1, \quad \text{by IH.}
\]
\[
\leq 4 \left( \frac{k + 1}{2} \right)^3 + k + 1, \quad \text{by def of floor.}
\]
\[
= 4 \left( \frac{(k + 1)^3}{2^3} \right) + k + 1, \quad \text{by algebra.}
\]
\[
= \frac{(k + 1)^3}{2} + k + 1, \quad \text{by algebra.}
\]
\[
= \frac{(k + 1)((k + 1)^2 + 2)}{2}, \quad \text{by algebra.}
\]
\[
\leq \frac{(k + 1)((k + 1)^2 + (k + 1)^2)}{2}, \quad \text{because } (k + 1)^2 \geq 2.
\]
\[
= (k + 1)^3, \quad \text{by algebra}
\]

Thus, since the base case and induction step hold, the \(P(n)\) is true for \(n \geq 3\).
3. [? points]
Let \( \Sigma = \{0, 1, 2\} \).
Consider \( L = \{w \in \Sigma^* : \text{Every 1 in the string has at least one 0 before and after it}\} \).

a) Give a regular expression that represents \( A \).

\[ \text{Solution: } (0 \cup 2)^*(0(0 \cup 1 \cup 2)^*0)(0 \cup 2)^* \]

b) Give a DFA that recognizes \( A \).

\[ \text{Solution: Omitted.} \]

c) Give a CFG that generates \( A \).

\[ \text{Solution:} \]
\[
S \rightarrow \varepsilon \mid 0S \mid 2S \mid 0ST \\
T \rightarrow 1R0S \\
R \rightarrow \varepsilon \mid 0R \mid 1R \mid 2R
\]
4. [? points]
Consider the following CFG: $S \rightarrow SS \mid S1 \mid S01$. Another way of writing the recursive definition of this set, $Q$, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then $w$ has at least as many 1’s as 0’s.

**Solution:** We go by structural induction on $w$. Let $P(w)$ be “$\#_0(w) \leq \#_1(w)$” for $w \in \Sigma^*$.

**Base Case.** When $w = \varepsilon$, note that $\#_0(\varepsilon) = 0 = \#_1(\varepsilon)$. So, the claim is true.

**Induction Hypothesis.** Suppose $P(w), P(v)$ are true for some $w, v$ generated by the grammar.

**Induction Step 1.** Note that $\#_0(w1) = \#_0(w) \leq \#_1(w) + 1 = \#_1(w1)$ by IH, and $\#_0(w01) = \#_0(w) + 1 \leq \#_1(w) + 1 = \#_1(w01)$ by IH.

**Induction Step 2.** Note that $\#_0(wv) = \#_0(w) + \#_0(v) \leq \#_1(w) + \#_1(v)$ by IH.

Since the claim is true for all recursive rules, the claim is true for all strings generated by the grammar.
5. [? points]
For each of the following answer True or False and give a short explanation of your answer.

- Any subset of a regular language is also regular.

  *Solution:* False. Consider \( \{0, 1\}^* \) and \( \{0^n1^n : n \geq 0\} \). Note that the first is regular and the second is irregular, but the second is a subset of the first.

- The set of programs that loop forever on at least one input is decidable.

  *Solution:* False. If we could solve this problem, we could solve HaltNoInput. Intuitively, a program that solves this problem would have to try all inputs, but, since the program might infinite loop on some of them, it won’t be able to.

- If \( R \subseteq A \) for some set \( A \), then \( A \) is uncountable.

  *Solution:* True. Diagonalization would still work; alternatively, if \( A \) were countable, then we could find an onto function between \( \mathbb{N} \) and \( \mathbb{R} \) by skipping all the elements in \( A \) that aren’t in \( R \).

- If the domain of discourse is people, the logical statement

  \[
  \exists x \ (P(x) \rightarrow \forall y \ (x \neq y \rightarrow \neg P(y)))
  \]

  can be correctly translated as “There exists a unique person who has property \( P \).”

  *Solution:* False. Any \( x \) for which \( P(x) \) is false makes the entire statement true. This is not the same as there existing a unique person with property \( P \).

- \( \exists x \ (\forall y \ P(x, y)) \rightarrow \forall y \ (\exists x \ P(x, y)) \) is true regardless of what predicate \( P \) is.

  *Solution:* True. The left part of the implication is saying that there is a single \( x \) that works for all \( y \); the right one is saying that for every \( y \), we can find an \( x \) that depends on it, but the single \( x \) that works for everything will still work.
6. [?] points
The following is the graph of a binary relation $R$.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

a) Draw the transitive-reflexive closure of $R$.

Solution: Omitted.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \land X \subseteq Y\}$.
Recall that $R$ is antisymmetric iff $(a, b) \in R \land a \neq b \rightarrow (b, a) \notin R$.
Prove that $S$ is antisymmetric.

Solution: Suppose $(a, b) \in S$ and $a \neq b$. Then, by definition of $S$, $a \subset b$ and there is some $x \in b$ where $x \notin a$ (since they aren't equal). Then, $(b, a) \notin S$, because $b \not\subseteq a$, because $x \in b$ and $x \notin a$. So, $S$ is antisymmetric.
7. [? points]
Convert the following NFA into a DFA using the algorithm from lecture.

Solution: Omitted.
8. [? points]
Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions $i$ where $i \mod 3 = 0$. 
Solution: Omitted.