Name: ________________________________

UW ID: ________________________________

Instructions:

- Closed book, closed notes, no cell phones, no calculators.
- You have 110 minutes to complete the exam.
- Answer all problems on the exam paper.
- If you need extra space use the back of a page.
- Problems are not of equal difficulty; if you get stuck on a problem, move on.
- You may tear off the last two pages of equivalence and inference rules. These must be handed in at the end but will not be graded.
- It may be to your advantage to read all the problems before beginning the exam.

Score Table Here
1. [? points]
Let \( \Sigma = \{0, 1\} \). Prove that the language \( L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\} \) is irregular.
2. [? points]
Define

\[ T(n) = \begin{cases} 
  n & \text{if } n = 0, 1 \\
  4T(\lfloor \frac{n}{2} \rfloor) + n & \text{otherwise}
\end{cases} \]

Prove that \( T(n) \leq n^3 \) for \( n \geq 3 \).
3. [?] points
Let $\Sigma = \{0, 1, 2\}.$
Consider $L = \{ w \in \Sigma^* : \text{Every 1 in the string has at least one 0 before and after it} \}.$

a) Give a regular expression that represents $A.$

b) Give a DFA that recognizes $A.$

c) Give a CFG that generates $A.$
4. [? points]
Consider the following CFG: $S \rightarrow SS \mid S1 \mid S01$. Another way of writing the recursive definition of this set, $Q$, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then $w$ has at least as many 1’s as 0’s.
5. [? points]
For each of the following answer True or False and give a short explanation of your answer.

- Any subset of a regular language is also regular.

- The set of programs that loop forever on at least one input is decidable.

- If $\mathbb{R} \subseteq A$ for some set $A$, then $A$ is uncountable.

- If the domain of discourse is people, the logical statement
  \[ \exists x \ (P(x) \rightarrow \forall y \ (x \neq y \rightarrow \neg P(y)) \]
  can be correctly translated as “There exists a unique person who has property $P$”.

- $\exists x \ (\forall y \ P(x, y)) \rightarrow \forall y \ (\exists x \ P(x, y))$ is true regardless of what predicate $P$ is.
6. [? points]
The following is the graph of a binary relation $R$.

![Graph of a binary relation]

a) Draw the transitive-reflexive closure of $R$.

![Transitive-reflexive closure of $R$]

b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \land X \subseteq Y\}$.
Recall that $R$ is antisymmetric iff $((a, b) \in R \land a \neq b) \rightarrow (b, a) \not\in R$.
Prove that $S$ is antisymmetric.
7. [? points]
Convert the following NFA into a DFA using the algorithm from lecture.

\[
\begin{array}{c}
q_0 \quad 1 \quad q_1 \\
\quad \quad 0 \quad \quad 1 \\
\quad \quad \quad \quad \epsilon \\
q_1 \quad 1 \quad q_2 \\
\end{array}
\]
8. [? points]
Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions $i$ where $i \mod 3 = 0$. 