CSE 311: Foundations of Computing

Fall 2014
Lecture 15: Strong Induction

Administrivia

- Midterm is in a week in lecture!
  - We will put out a practice exam + practice questions later today!
  - There will be three review sessions (one on Thursday, one on Saturday, and one on Sunday)

---

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be “$3^n \geq n^2$” for all $n \geq 3$.
We go by induction on $n$.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, $P(3)$ is true.

Induction Hypothesis:

Suppose $P(k)$ is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.
Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.
Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.
So, continuing from above:

$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 + 2k + 1 = (k+1)^2$

Since this is exactly $P(k+1)$, we’ve shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.

---

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.
Let $P(n)$ be “$2n^3 + 2n - 5 \geq n^2$” for all $n \geq 2$.
We go by induction on $n$.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, $P(0)$ is true.

Induction Hypothesis:

Suppose $P(n)$ is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that...

\[
\begin{align*}
(n+1)^2 & \leq n^2 + 2n + 1 \\
& \leq (2n^3 + 2n - 5) + 2n + 1 \quad \text{(by IH)} \\
& \leq (2n^3 + 4n + 1) - 5 \quad \text{(Re-arranging)} \\
& \leq (2n^3 + 6n^2 + 6n + 2) - 5 \quad (4n + 1 \leq 6n + 6n^2 + 2) \\
& \leq 2(n+1)^3 - 5 \quad \text{(Factoring)} \\
& \leq 2(n+1)^3 + 2n - 5 \quad (0 \leq 2n)
\end{align*}
\]

Since this is exactly $P(k+1)$, we’ve shown $P(k) \rightarrow P(k+1)$
Thus, $P(n)$ is true for all $n \geq 3$, by induction.
**Strong Induction**

\[ P(0) \]
\[
\forall k \left( (P(0) \land P(1) \land P(2) \land \cdots \land P(k)) \implies P(k + 1) \right)
\]
\[
\therefore \forall n P(n)
\]

*Follows from ordinary induction applied to*

\[ Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n) \]

---

**Strong Induction English Proof**

1. By induction we will show that \( P(n) \) is true for every \( n \geq 0 \)
2. Base Case: Prove \( P(0) \)
3. Inductive Hypothesis:
   Assume that for some arbitrary integer \( k \geq 0 \), \( P(j) \) is true for every \( j \) from 0 to \( k \)
4. Inductive Step:
   Prove that \( P(k + 1) \) is true using the Inductive Hypothesis (that \( P(j) \) is true for all values \( \leq k \))
5. Conclusion: Result follows by induction

---

**Every integer at least 2 is the product of primes**

We go by strong induction. Let \( P(n) \) be “\( n \) can be expressed as a product of primes” for \( n \geq 2 \).

**Base Case:**
Note that 2 is prime; so, we can express it as “2” which is a product of primes.

**Induction Hypothesis:**
Suppose \( P(2) \land P(3) \land \cdots \land P(k) \) is true for some \( k \geq 2 \).

**Induction Step:**
We go by cases.
Suppose \( k+1 \) is prime. Then, “\( k+1 \)” is a product of primes.
Suppose \( k+1 \) is composite. Then, \( k+1 = ab \) for some \( a \) and \( b \) such that \( 1 < a, b < k+1 \).
By our IH, we know \( a = p_1p_2\cdots p_n \) and \( b = q_1q_2\cdots q_m \).
So, \( k+1 = ab = p_1p_2\cdots p_nq_1q_2\cdots q_m \), which is a product of primes.

Thus, our claim is true for \( n \geq 2 \) by strong induction.

---

**Recursive Definitions of Functions**

- \( F(0) = 0; F(n + 1) = F(n) + 1 \) for all \( n \geq 0 \)
- \( G(0) = 1; G(n + 1) = 2 \times G(n) \) for all \( n \geq 0 \)
- \( 0! = 1; (n + 1)! = (n + 1) \times n! \) for all \( n \geq 0 \)
- \( H(0) = 1; H(n + 1) = 2^H(n) \) for all \( n \geq 0 \)
Fibonacci Numbers

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2 \]

Bounding the Fibonacci Numbers

Theorem: \[ f_n < 2^n \text{ for all } n \geq 2. \]