Fall 2014
Lecture 14: Induction
Mathematical Induction

Method for proving statements about all natural numbers

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to use the special structure of the naturals to prove things more easily

- Particularly useful for reasoning about programs!

```java
for(int i=0; i < n; n++) {
    ...}
```
- Show \( P(i) \) holds after \( i \) times through the loop

```java
public int f(int x) {
    if (x == 0) {
        return 0;
    }
    else {
        return f(x - 1);
    }
}
```
- \( f(x) = x \) for all values of \( x \geq 0 \) naturally shown by induction.
Prove for all $k > 0$, $n^k$ even $\rightarrow$ n even

Let $k > 0$ be arbitrary. We go by contrapositive. Suppose that $n$ is odd. We know that if $a, b$ are odd, then $ab$ is also odd.

So,

$$(\ldots \bullet (n \bullet n) \bullet n) \bullet \ldots \bullet n = n^k$$

(k times)

Those “…”s are a problem! We’re trying to say “we can use the same argument over and over”... We should use induction instead.
**Induction Is A Rule of Inference**

**Domain: Natural Numbers**

\[
P(0) \quad \forall k \ (P(k) \rightarrow P(k + 1))
\]

\[\therefore \forall n \ P(n)\]

**How does this technique prove P(5)?**

By Induction:
- \[P(0) \rightarrow P(1)\]
- \[P(1) \rightarrow P(2)\]
- \[P(2) \rightarrow P(3)\]
- \[P(3) \rightarrow P(4)\]
- \[P(4) \rightarrow P(5)\]

To Prove:
- \(P(0)\)
- \(P(1)\)
- \(P(2)\)
- \(P(3)\)
- \(P(4)\)
- \(P(5)\)

First, we prove \(P(0)\).

Since \(P(n) \rightarrow P(n+1)\) for all \(n\), we have \(P(0) \rightarrow P(1)\).

Since \(P(0)\) is true and \(P(0) \rightarrow P(1)\), by Modus Ponens, \(P(1)\) is true.

Since \(P(n) \rightarrow P(n+1)\) for all \(n\), we have \(P(1) \rightarrow P(2)\).

Since \(P(1)\) is true and \(P(1) \rightarrow P(2)\), by Modus Ponens, \(P(2)\) is true.
Using The Induction Rule In A Formal Proof

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k + 1)) \]

\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \)
2. Let \( k \) be an arbitrary integer \( \geq 0 \)
   3. Assume that \( P(k) \) is true
   4. ...
   5. Prove \( P(k+1) \) is true

6. \( P(k) \rightarrow P(k+1) \) \hspace{2cm} \text{Direct Proof Rule}
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \) \hspace{2cm} \text{Intro } \forall \text{ from 2-6}
8. \( \forall n \ P(n) \) \hspace{2cm} \text{Induction Rule 1&7}
What can we say about $1 + 2 + 4 + 8 + ... + 2^n$

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

- Can we describe the pattern?
  - $1 + 2 + 4 + ... + 2^n = 2^{n+1} - 1$
Proving $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

• We could try proving it normally...
  – We want to show that $1 + 2 + 4 + \ldots + 2^n = 2^{n+1}$.
  – So, what do we do now? We can sort of explain the pattern, but that’s not a proof...

• We could prove it for $n=1$, $n=2$, $n=3$, ... (individually), but that would literally take forever...
Instead, Let’s Use Induction

\[ P(0) \]
\[ \forall k \ (P(k) \rightarrow P(k+1)) \]

\[ \therefore \forall n \ P(n) \]

1. Prove \( P(0) \)  
2. Let \( k \) be an arbitrary integer \( \geq 0 \)  
3. Assume that \( P(k) \) is true  
4. ...  
5. Prove \( P(k+1) \) is true  
6. \( P(k) \rightarrow P(k+1) \)  
7. \( \forall k \ (P(k) \rightarrow P(k+1)) \)  
8. \( \forall n \ P(n) \)

**Base Case**

**Inductive Hypothesis**

**Inductive Step**

**Conclusion**

**Direct Proof Rule**

**Intro \( \forall \) from 2-6**

**Induction Rule 1&7**
5 Steps To Inductive Proofs In English

Proof:
1. “We will show that P(n) is true for every \( n \geq 0 \) by Induction.”
2. “Base Case:” Prove P(0)
3. “Inductive Hypothesis:”
   Assume P(k) is true for some arbitrary integer \( k \geq 0 \)
4. “Inductive Step:” Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don’t assume P(k+1) !!)
5. “Conclusion: Result follows by induction”
Proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$.

Let $P(n)$ be “$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$”. We will show $P(n)$ is true for all natural numbers by induction.

**Base Case (n=0):**

$$2^0 = 1 = 2 - 1 = 2^{0+1} - 1$$

**Induction Hypothesis:**

Suppose that $P(k)$ is true for some arbitrary $k \geq 0$.

**Induction Step:**

**WTS:** Show $P(k+1)$ (i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$)

1. $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ (by IH!)
2. Adding $2^{k+1}$ to both sides, we get:
   $$1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$
3. Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.
4. So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.
Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Let $P(n)$ be $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

We go by induction on $n$.

**Base Case:**
When $n=1$, we have $1 = 1(2)/2$. So, $P(0)$ is true.

**Induction Hypothesis:**
Suppose $P(k)$ is true for some $k \geq 1$.

**Induction Step:**
We know $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, by the IH. We want to prove $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Note that $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$ by the IH.

And $\sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2}$ is exactly $P(k+1)$.

The claim follows for all $n \geq 1$, by induction.
Another Pattern

• $20 - 1 = 1 - 1 = 0 = 3 \cdot 0$
• $22 - 1 = 4 - 1 = 3 = 3 \cdot 1$
• $24 - 1 = 16 - 1 = 15 = 3 \cdot 5$
• $26 - 1 = 64 - 1 = 63 = 3 \cdot 21$
• $28 - 1 = 256 - 1 = 255 = 3 \cdot 85$
• ...
Another Example

We want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let $P(n)$ be “$2^{2n} - 1 = 3k$ for some integer $k$” for all $n \geq 0$.

We go by induction.

**Base Case:** When $n = 0$, note that $2^0 - 1 = 0 = 3k$. So, $P(0)$ is true.

**Induction Hypothesis:** Suppose $P(k)$ is true for some $k \geq 0$.

**Induction Step:** Note that $2^{2(k+1)} - 1 = (2^{2k})(2^2) - 1$. By the IH, $2^{2k} - 1 = 3j$ for some $j$. So, $2^{2(k+1)} - 1 = (2^{2k})(2^2) - 1 = (3j + 1)(2^2) - 1 = 12j + 3 = 3(4j + 1)$. It follows that there is some $r$ (namely, $r = 4j + 1$) such that $2^{2(k+1)} - 1 = 3r$.

Since $P(0)$ is true, and $P(k) \rightarrow P(k+1)$ for all $k \geq 0$, the claim is true by induction.