CSE 311: Foundations of Computing I  
Section: Relations, CFGs, and DFAs Solutions

**CFGs**
Construct CFGs for the following languages:

(a) All binary strings that end in 00.

*Solution:*

\[
S \rightarrow 0S | 1S | 00
\]

(b) All binary strings that contain at least three 1's.

*Solution:*

\[
\begin{align*}
S & \rightarrow 0S | 1T \\
T_1 & \rightarrow 0T_1 | 1T_2 \\
T_2 & \rightarrow 0T_2 | 1T_3 \\
T_3 & \rightarrow 0T_3 | 1T_3 | \varepsilon
\end{align*}
\]

(c) All binary strings with an equal number of 1's and 0's.

*Solution:*

\[
\begin{align*}
S & \rightarrow 0S1S | 1S0S | \varepsilon \\
S & \rightarrow SS | 0S1 | 1S0 | \varepsilon
\end{align*}
\]

**Relations**
(a) Draw the transitive-reflexive closure of \{ (1, 2), (2, 3), (3, 4) \}.

*Solution:*
(b) Suppose that $R$ is reflexive. Prove that $R \subseteq R^2$.

Solution: Suppose $(a, b) \in R$. Since $R$ is reflexive, we know $(b, b) \in R$ as well. Since there is a $b$ such that $(a, b) \in R$ and $(b, b) \in R$, it follows that $(a, b) \in R^2$. Thus, $R \subseteq R^2$.

(c) Consider the relation $R = \{(x, y) : x = y + 1\}$ on $\mathbb{N}$. Is $R$ reflexive? Transitive? Symmetric? Anti-symmetric?

Solution: It isn’t reflexive, because $1 \neq 1 + 1$; so, $(1, 1) \not\in R$. It isn’t symmetric, because $(2, 1) \in R$ (because $2 = 1 + 1$), but $(1, 2) \not\in R$, because $1 \neq 2 + 1$. It isn’t transitive, because note that $(3, 2) \in R$ and $(2, 1) \in R$, but $(3, 1) \not\in R$. It is anti-symmetric, because consider $(x, y) \in R$ such that $x \neq y$. Then, $x = y + 1$ by definition of $R$. However, $(y, x) \not\in R$, because $y = x - 1 \neq x + 1$.

(d) Consider the relation $S = \{(x, y) \mid x^2 = y^2\}$ on $\mathbb{R}$. Prove that $S$ is reflexive, transitive, and symmetric.

Solution: Consider $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x, x) \in R$; so, $R$ is reflexive.

Consider $(x, y) \in R$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y, x) \in R$. So, $R$ is symmetric. Suppose $(x, y) \in R$ and $(y, z) \in R$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x, z) \in R$. So, $R$ is transitive.

**DFAs**

Construct a DFA for the language of all binary strings, where $\Sigma = \{0, 1, 2\}$.

Solution: Omitted.