Section 4 Worksheet

Solutions
(1) More on Sets

Prove that \( A \subseteq B \iff B' \subseteq A' \).

**Proof:** (\( \rightarrow \)) Let \( A \subseteq B \). Assume an element \( x \) is a member of \( B' \). (We want to show \( x \in A' \).)

Then \( x \notin B \) by definition of set complements.

Thus \( x \notin A \) because \( A \subseteq B \) by assumption, and since \( x \notin B \rightarrow x \notin A \). (Contrapositive of the definition of subset.)

\[ \therefore \] \( x \in A' \) by definition of set complement.

So \( x \in B' \rightarrow x \in A' \), and thus \( B' \subseteq A' \) by def. of subset.
Prove that $A \subseteq B \leftrightarrow B' \subseteq A'$.

($\leftarrow$) Let $B' \subseteq A'$. Assume an element $x$ is a member of $A$. (We want to show $x \in B$.)

Then $x \notin A'$ by definition of set complements.

Thus $x \notin B'$ because $B' \subseteq A'$ by assumption, and since $x \notin A' \rightarrow x \notin B'$. (Contrapositive of the definition of subset.)

$\therefore x \in B$ by definition of set complement.

So $x \in A \rightarrow x \in B$, and thus $A \subseteq B$ by def. of subset.
(1) More on Sets

Prove that $A \subseteq B \leftrightarrow B' \subseteq A'$.

We have shown $A \subseteq B \rightarrow B' \subseteq A'$ and $B' \subseteq A' \rightarrow A \subseteq B$, thus we have proven $A \subseteq B \leftrightarrow B' \subseteq A'$. ■
(2) Functions

\[ A = \{x : x \in \mathbb{R}, \ x \geq 1\} \]
\[ B = \{x : x \in \mathbb{R}, \ 0 \leq x \leq 1\} \]
\[ C = \{x : x \in \mathbb{R}, \ -1 \leq x \leq 1\} \]

(i) \[ f : A \rightarrow B, \ f(x) = \frac{1}{x} \]

One-to-one, but not onto.
(0 \in B, but we can never get 0 by plugging in any value in our domain.)
(2) Functions

A = \{x : x \in \mathbb{R}, x \geq 1\}
B = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}
C = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}

(ii) \ f : B \rightarrow C, f(x) = x^2

One-to-one, but not onto.
(-1 \in C, but we can never get -1 by plugging in any value in our domain.)
(2) Functions

A = \{x : x \in \mathbb{R}, x \geq 1\}
B = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}
C = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}

(iii) \quad f : B \to B, f(x) = x^2

Both one-to-one and onto.
(No negatives to worry about in this case, so we don’t have the same problem as before for onto. One-to-one because no two values in the domain produce the same output.)
(2) Functions

A = \{x : x \in \mathbb{R}, x \geq 1\}
B = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}
C = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}

(iv) \ f : C \rightarrow B, f(x) = x^2

Onto, but not one-to-one.
(-1 and 1 are both in domain, both produce output of 1.)
(3) Modular Arithmetic

Find an integer $a$ such that:

(i) $a \equiv 43 \pmod{23}, \ -22 \leq a \leq 0$

$a = -3$ (we can check by seeing that $23 \mid (43-(-3)))$

Def: Let $a, b$ be integers, and $m$ be a positive integer. Then $a \equiv b \pmod{m} \iff m \mid (a-b)$. 
(3) Modular Arithmetic

Find an integer $a$ such that:

(i) $a \equiv 17 \pmod{29}$, $-14 \leq a \leq 14$

$a = -12$

(Check: $29 \mid -29 \checkmark$)
(3) Modular Arithmetic

Find an integer $a$ such that:
(i) $a \equiv -11 \pmod{21}$, $90 \leq a \leq 110$

$a = 94$

(Check: $21 \mid 105 \checkmark$)