Problem 1:
Suppose that the sets $A$, $B$, and $C$ have 4, 6, and 8 elements respectively. For each of the statements below, indicate whether it is certainly true, or certainly false, or can be either true or false. Briefly justify your answers:
(a) $A \cup B$ has exactly 10 elements.
(b) $A \cap B$ has at most 4 elements.
(c) If $A \cup B$ has $m$ elements and $A \cap B$ has $n$ elements, then $m + n$ is always equal to 10.
(d) $A \cup B$ has at most as many elements as $A \cup C$.
(e) $(A \oplus B) \oplus (B \oplus C) \oplus (A \oplus B \oplus C)$ has exactly 6 elements; $\oplus$ denotes the symmetric difference of two sets.

Problem 2:
Consider the following functions from the positive integer numbers (including zero) to the positive integers (including zero). For each of the functions below, indicate the following: (i) its domain, (ii) its range, (iii) whether the function is one-to-one, (iv) whether the function is onto. Briefly justify your answers.
(a) $f(n) = n + 1$ if $n$ is even, and $f(n) = n - 1$ if $n$ is odd.
(b) $f(n) = n/2$ if $n$ is even, $f(n) = (3n + 1)/2$ if $n$ is odd.
(c) $f(n) = 2^n$.
(d) $f(n) = \lfloor \log(n) \rfloor$. The logarithm is in base two. If $x$ is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$; for example $\lfloor 4.7 \rfloor = 4$ and $\lfloor 4.0 \rfloor = 4$.
(e) $f(n) = \left\lfloor \frac{10n-1}{n-1} \right\rfloor$.

Problem 3:
Prove that if $n$ is an integer then $n^2 \mod 5$ is either 0, 1, or 4.

Problem 4:
Let $a, b$ be integers and $c, n$ be positive integers. Prove that if $a \equiv b \pmod{n}$ then $ac \equiv bc \pmod{cn}$.

Problem 5:
Show that a positive integer is divisible by 11 if and only if the difference of the sum of its decimal digits in even-numbered positions and the sum of the decimal digits in its odd-numbered positions is divisible by 11.
Problem 6:
For each $a \in \{1, \ldots, 12\}$ determine the smallest integer $k \geq 1$ such that $a^k \mod 13 = 1$.

Problem 7:
Compute $43^{148} \mod 1000$ using the fast modular exponentiation algorithm. Show your intermediate results. (Hint: this problem only requires 9 multiplications.)

Extra Credit 8:
The number 2011 is a prime number.

1. Compute $3 \times 1341 \mod 2011$ (you can use a calculator).
2. Compute $4 \times 503 \mod 2011$ (you can use a calculator).
3. Compute $5 \times 1609 \mod 2011$ (you can use a calculator).
4. Compute $2010! \mod 2011$; this is the same as $1 \times 2 \times \cdots \times 2010 \mod 2011$ (you don’t need a calculator here).