mathematical induction

Method for proving statements about all integers $n \geq 0$.

- Part of sound logical inference that applies only in the domain of integers
  - Not like scientific induction which is more like a guess from examples
- Particularly useful for reasoning about programs since the statement might be “after $n$ times through this loop, property $P(n)$ holds”

finding a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- ...

announcements

Reading assignment

Induction

5.1-5.2, 7th edition
4.1-4.2, 6th edition
how do you prove it?

Want to prove $3 \mid 2^n - 1$ for all integers $n \geq 0$

$-n = 0$
$-n = 1$
$-n = 2$
$-n = 3$
$- ...$

induction as a rule of Inference

Domain: integers $\geq 0$

$P(0)$
$\forall \ k \ (P(k) \rightarrow P(k+1))$

$\therefore \ \forall \ n \ P(n)$

using the induction rule in a formal proof

$P(0)$
$\forall \ k \ (P(k) \rightarrow P(k+1))$

$\therefore \ \forall \ n \ P(n)$

1. Prove $P(0)$
2. Let $k$ be an arbitrary integer $\geq 0$
   3. Assume that $P(k)$ is true
   4. ...
   5. Prove $P(k+1)$ is true
6. $P(k) \rightarrow P(k+1)$ Direct Proof Rule
7. $\forall \ k \ (P(k) \rightarrow P(k+1))$ Intro $\forall$ from 2-6
8. $\forall \ n \ P(n)$ Induction Rule 1&7

using the induction rule in a formal proof

$P(0)$
$\forall \ k \ (P(k) \rightarrow P(k+1))$

$\therefore \ \forall \ n \ P(n)$

1. Prove $P(0)$ \textbf{Base Case}
2. Let $k$ be an arbitrary integer $\geq 0$ \textbf{Inductive Hypothesis}
   3. Assume that $P(k)$ is true
   4. ...
   5. Prove $P(k+1)$ is true \textbf{Inductive Step}
6. $P(k) \rightarrow P(k+1)$ Direct Proof Rule
7. $\forall \ k \ (P(k) \rightarrow P(k+1))$ Intro $\forall$ from 2-6
8. $\forall \ n \ P(n)$ Induction Rule 1&7 \textbf{Conclusion}
5 steps to inductive proofs in english

Proof:
1. “By induction we will show that P(n) is true for every n ≥ 0.”
2. “Base Case:” Prove P(0)
3. “Inductive Hypothesis:”
   Assume P(k) is true for some arbitrary integer k ≥ 0
4. “Inductive Step:” Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don’t assume P(k+1) !!)
5. “Conclusion: Result follows by induction”

induction example

Want to prove 3 | 2^{2n} - 1 for all n ≥ 0.

geometric sum

1 + 2 + 4 + ⋯ + 2^n = 2^{n+1} - 1 for all n ≥ 0
For all $n \geq 0$:

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

**sum of first $n$ numbers**

For all $n \geq 1$:

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

**harmonic numbers**

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} = \sum_{i=1}^{m} \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$. 
Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$.

## checkerboard tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:

Follows from ordinary induction applied to

\[
(P(0) \land P(1) \land P(2) \land \cdots \land P(k)) \rightarrow P(k+1)
\]

\[\therefore \forall n \ P(n)\]

Follows from ordinary induction applied to

\[Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)\]

### strong induction english proofs

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every $j$ from 0 to $k$
4. Inductive Step: Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction
every integer $\geq 2$ is the product of primes

recursive definitions of functions

- $F(0) = 0; F(n + 1) = F(n) + 1$ for all $n \geq 0$
- $G(0) = 1; G(n + 1) = 2 \cdot G(n)$ for all $n \geq 0$
- $0! = 1; (n + 1)! = (n + 1) \cdot n!$ for all $n \geq 0$
- $H(0) = 1; H(n + 1) = 2^{H(n)}$ for all $n \geq 0$

fibonacci numbers

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$
Bounding the Fibonacci numbers

Theorem: \( \frac{2}{5} \leq f_n < 2^n \) for all \( n \geq 2 \).

Theorem: \( 2^{n/2-1} \leq f_n < 2^n \) for all \( n \geq 2 \).