2 More on sets.

Prove that \( A \subseteq B \iff \bar{B} \subseteq \bar{A} \).

**Proof.** (For a biconditional statement \( P \iff Q \), we must show both that \( P \rightarrow Q \) and \( Q \rightarrow P \) in order to complete the proof.)

(\( \rightarrow \)) Let \( A \subseteq B \), and suppose \( x \in \bar{B} \).
Then \( x \notin B \) by definition of set complements.
Since \( A \subseteq B \), then \( \forall y(y \in A \rightarrow y \in B) \) [contrapositive], so it follows that \( x \notin A \).
Therefore \( x \in \bar{A} \) by def. of set complements.
Since we have shown that \( x \in \bar{B} \rightarrow x \in \bar{A} \), then \( \bar{B} \subseteq \bar{A} \) by definition of subset.

(\( \leftarrow \)) Let \( \bar{B} \subseteq \bar{A} \), and suppose \( x \in A \). By a symmetrical argument, \( x \in B \). Thus \( A \subseteq B \). \( \square \)

3 Memories of functions.

For all functions and mappings below, state whether they are injective (one-to-one), surjective (onto), or bijective (both) over the following sets:

\[
\begin{align*}
A &= \{ x | x \in \mathbb{R}, x \geq 1 \} \\
B &= \{ x | x \in \mathbb{R}, 0 \leq x \leq 1 \} \\
C &= \{ x | x \in \mathbb{R}, -1 \leq x \leq 1 \}
\end{align*}
\]

1. \( f: A \rightarrow B \), \( f(x) = \frac{1}{x} \)
   **Answer:** Injective, but not surjective (\( 0 \in B \), but \( \frac{1}{x} \neq 0 \) \( \forall x \in A \).)

2. \( f: B \rightarrow C \), \( f(x) = x^2 \)
   **Answer:** Injective, but not surjective (\( -1 \in C \), but \( x^2 \neq -1 \) \( \forall x \in B \).)

3. \( f: B \rightarrow B \), \( f(x) = x^2 \)
   **Answer:** Both one-to-one and onto, so bijective. (No negatives to worry about in this case, so we don’t have the same problem as 2 for surjective or the same problem as 4 for injective.)

4. \( f: C \rightarrow B \), \( f(x) = x^2 \)
   **Answer:** Surjective, but not injective. (\( f(-1) = f(1) = 1 \), but \( -1 \neq 1 \))
4 Modular Arithmetic.

Find $a \in \mathbb{Z}$ such that:

1. $a \equiv 43 \pmod{23}, \ -22 \leq a \leq 0$
   \[\text{Answer: } a = -3 \text{ (we can check by seeing that } 23|(43 - (-3))\]

2. $a \equiv 17 \pmod{29}, \ -14 \leq a \leq 14$
   \[\text{Answer: } a = -12 \text{ (we can check by seeing that } 29|(17 - (-12))\]

3. $a \equiv -11 \pmod{21}, \ 90 \leq a \leq 110$
   \[\text{Answer: } a = 94 \text{ (we can check by seeing that } 21|(94 - (-11))\]