1 Using Strong Induction
Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.


2 Recursive functions
Find \( f(1), f(2), f(3), \) and \( f(4) \) if \( f(n) \) is defined recursively by \( f(0) = 1 \) and for \( n = 0, 1, 2, \ldots \)

a) \( f(n + 1) = f(n) + 2 \)
Answer: \( f(1) = 3, f(2) = 5, f(3) = 7, f(4) = 9 \)

b) \( f(n + 1) = 3f(n) \)
Answer: \( f(1) = 3, f(2) = 9, f(3) = 27, f(4) = 81 \)

c) \( f(n + 1) = 2^{f(n)} \)
Answer: \( f(1) = 2, f(2) = 4, f(3) = 16, f(4) = 2^{16} \)

d) \( f(n + 1) = f(n)^2 + f(n) + 1 \)
Answer: \( f(1) = 3, f(2) = 13, f(3) = 183, f(4) = 33,673 \)
3 Recursive proof

Prove that \( f_0 f_1 + f_1 f_2 + \ldots + f_{2n-1} f_{2n} = f_{2n}^2 \) where \( n \) is a positive integer and \( f_n \) is the \( n \)th Fibonacci number.

Proof. (By induction)

**Base Case:** Let \( n = 1 \).
Then \( f_0 f_1 + f_1 f_2 = (0)(1) + (1)(1) = 1 \) and \( f_2 = (1)(1) = 1 \), therefore \( f_0 f_1 + f_1 f_2 = f_2^2 \).

**Inductive Step:** Assume \( f_0 f_1 + f_1 f_2 + \ldots + f_{2k-1} f_{2k} = f_{2k}^2 \).
Show that \( f_0 f_1 + \ldots + f_{2(k+1)-1} f_{2(k+1)} = f_{2(k+1)}^2 \).

So, starting with the left hand side of our equation (and simplifying the subscripts), we have:
\[
\begin{align*}
f_0 f_1 + f_1 f_2 + \ldots + f_{2k+1} f_{2k+2} &= (f_0 f_1 + f_1 f_2 + \ldots + f_{2k-1} f_{2k}) + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\
&= (f_{2k}^2) + f_{2k} (f_{2k+1} + f_{2k+1} f_{2k+2}) \quad \text{(by inductive hypothesis)} \\
&= f_{2k} f_{2k} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \quad \text{(expanding squared term)} \\
&= f_{2k} (f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} \quad \text{(by distributive property)} \\
&= f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} \quad \text{(by def. of Fib. numbers)} \\
&= f_{2k+2} f_{2k+2} \quad \text{(by distributive property)} \quad \text{(by def. of Fib numbers)} \\
&= f_{2(k+1)}^2 \quad \text{(by distributive property)}
\end{align*}
\]

This is what we were trying to show, thus we have satisfied our inductive step and proved the statement for all integers \( n \geq 1 \). \( \square \)