CSE 311 Sample final problems.

Problem 1:

2. (a) Draw the state diagram of an NFA \( M \) that recognizes the language \( a^*b(b \cup ab)^*a \). You don’t have to use any particular construction method but you should try to avoid unnecessary states.

Problem 2:

The epsilon transition is the same as the lambda transition we used in class.

3. Build a DFA equivalent to the following NFA using the “subset construction.” You only need to show states that are reachable from the start state of your DFA (but do not attempt to simplify further).

![State Diagram]

Problem 3:

2. (30 points) Define the language \( A = \{w \in \{0,1\}^* \mid \text{the number of 0's minus the number of 1's in } w \text{ is divisible by } 3\} \).

   (a) Construct a DFA with only three states that recognizes \( A \).

Problem 4:

Show that the following language cannot be recognized by a DFA:

\[ \{a^n b a^m b a^{m+n} \mid n, m \geq 1\} \]

Give a CFG for the language

\[ \{a^n b a^m b a^{m+n} \mid n, m \geq 1\} \]
Problem 5:

Consider the NFA \( N = (Q, \Sigma, \delta, q_0, F) \) with the following state diagram:

![NFA Diagram]

a. What states can \( N \) be in after reading:
   - the string 0?
   - the string 01?
   - the string 011?

b. Does \( N \) accept 011? Why or why not?

Problem 6:

Let \( L \) be the set of strings in \( \{a, b\}^* \) such that each a, if any, has two b’s immediately to its right. Give:

a) A finite automaton accepting \( L \).
   b) A regular expression denoting \( L \).
   c) A Context free grammar generating \( L \).

Problem 7:

In the land of Garbanzo, the unit of currency is the bean. They only have two coins, one worth 2 beans and the other worth 5 beans.

a. Give a recursive definition of the set of positive integers \( S \) such that \( x \) is in \( S \) if and only if one can make up an amount worth \( x \) beans using at most one 5-bean coin and any number of 2-bean coins.

b. Prove by strong induction that every integer \( \geq 4 \) is in \( S \). (You do NOT need a recursive proof here.)
Problem 8:

Let \( R \) be the relation \( \{ (1,2), (3,4), (1,3), (2,1) \} \) defined on the set \( \{1,2,3,4,5\} \).

- Draw the graph of \( R \).
- Draw the graph of \( R^2 \).
- Draw the graph of the reflexive-transitive closure of \( R \).

Problem 9:

For each \( n \geq 0 \) define \( T_n \), the 'complete 3-ary tree of height \( n \) ' as follows:

- \( T_0 \) is an undirected graph consisting of a single vertex called the root of \( T_0 \).
- For \( n \geq 0 \), \( T_{n+1} \) is an undirected graph consisting of a new vertex of degree 3 joined to the roots of 3 disjoint copies of \( T_n \). The new vertex is called the root of \( T_{n+1} \).

Prove by induction that for each \( n \geq 0 \), \( T_n \) has exactly \( \frac{3^{n+1} - 1}{2} \) vertices.

Problem 10:

Suppose \( R_1 \) and \( R_2 \) are transitive-reflexive relations on a set \( A \). Is the relation \( R_1 \) UNION \( R_2 \) necessarily a transitive-reflexive relation? Justify your answer

Problem 11:

- Every day, starting on day 0, one vampire arrives in Seattle from Transylvania and, starting on the day after its arrival, bites one Seatlite every day. People bitten become vampires themselves and live forever. New vampires also bite one person each day starting the next day after they were bitten. Let \( V_n \) be the number of vampires in Seattle on day \( n \). So, for example, \( V_0 = 1 \), \( V_1 = 3 \) (one that arrived from Transylvania on day 0, one that he bit on day 1, and another one that arrived from Transylvania on day 1), \( V_2 = 7 \) and so on. Write a recurrence relation for \( V_n \) that is valid for any \( n \geq 2 \).
- Prove by induction on \( n \) that \( V_n \leq 3^n \).
Problem 12:

Suppose that for all $n \geq 1$

$$g(n + 1) = \max_{1 \leq k \leq n} [g(k) + g(n + 1 - k) + 1]$$

and that $g(1) = 0$. Prove by induction that $g(n) = n - 1$ for all $n \geq 1$. 