Relations

Sections 8.1 & 8.5

Based on Rosen and slides by K. Busch

Relations and Their Properties

A binary relation $R$ from set $A$ to set $B$ is a subset of Cartesian product $A \times B$

Example: $A = \text{UW students}$ $B = \text{UW courses}$

$R = \{(a,b) \mid a \text{ is enrolled in } b\}$

Example: $A = \{0,1,2\}$ $B = \{a, b\}$

$R = \{(0,a), (0,b), (1,a), (2,b)\}$
A relation on set $A$ is a subset of $A \times A$

Example:

A relation on set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$$

More Examples

Relations over integers:

- $R = \{(a, b) \mid a > b\}$
- $R = \{(a, b) \mid a = b \text{ or } a = -b\}$
- $R = \{(a, b) \mid a \equiv b \mod m\}$ for positive integer $m > 1$
- $R = \{(a, b) \mid b = a + 1\}$
  
  (Actually a function)
Functions as Relations

$R = \{(a,b) \mid b = a + 1\}$  Relation over integers $\mathbb{Z}$

$f(a) = b = a + 1$  Function from $\mathbb{Z}$ to $\mathbb{Z}$

$f : \mathbb{Z} \rightarrow \mathbb{Z}$

Function from $A$ to $B$ assigns exactly one element from $B$ to each input from $A$

i.e., a function is a restricted type of relation where every $a$ in $A$ is in exactly one ordered pair $(a,b)$.

Reflexive relation $R$ on set $A$:

$\forall a \in A, \ (a,a) \in R$

Example: $A = \{1,2,3,4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (3,3), (4,3), (4,4)\}$
Symmetric relation $R$ :

$$(a, b) \in R \rightarrow (b, a) \in R$$

Example: $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (4,4)\}$

Antisymmetric relation $R$ :

$$(a, b) \in R \land (b, a) \in R \rightarrow a = b$$

Example: $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,2), (3,4), (4,4)\}$
Transitive relation $R$:

$$(a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R$$

Example: $A = \{1, 2, 3, 4\}$

$$R = \{(1,1), (1,2), (2,3), (3,4)(1,3), (1,4), (2,4)\}$$

Combining Relations

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$

$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$
Composite relation: \( S \circ R \)
\((a, b) \in S \circ R \iff \exists x : (a, x) \in R \land (x, b) \in S \)

Note: \((a, b) \in R \land (b, c) \in S \rightarrow (a, c) \in S \circ R \)

Example:
\[
R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}
\]
\[
S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}
\]
\[
S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}
\]

Power of relation: \( R^n \)
\[
R^1 = R \quad R^{n+1} = R^n \circ R
\]

Example: \( R = \{(1,1), (2,1), (3,2), (4,3)\} \)
\[
R^2 = R \circ R = \{(1,1), (2,1), (3,1)(4,2)\}
\]
\[
R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1)(4,1)\}
\]
\[
R^4 = R^3 \circ R = R^3
\]
Theorem: A relation \( R \) is transitive if and only if \( R^n \subseteq R \) for all \( n = 1,2,3,\ldots \)

Proof: 1. If part: \( R^2 \subseteq R \)

2. Only if part: use induction

1. If part: We will show that if \( R^2 \subseteq R \) then \( R \) is transitive

Assumption: \( R^2 \subseteq R \)

Definition of power: \( R^2 = R \circ R \)

Definition of composition:
\[ (a,b) \in R \land (b,c) \in R \rightarrow (a,c) \in R \circ R \]

Therefore, \( R \) is transitive
2. Only if part:

We will show that if $R$ is transitive then $R^n \subseteq R$ for all $n \geq 1$

Proof by induction on $n$

Inductive basis: $n = 1$

It trivially holds $R^1 = R \subseteq R$

Inductive hypothesis:

Assume that $R^k \subseteq R$

for all $1 \leq k \leq n$
Inductive step: We will prove $R^{n+1} \subseteq R$

Take arbitrary $(a, b) \in R^{n+1}$

We will show $(a, b) \in R$

\[
(a, b) \in R^{n+1} \\
\downarrow \quad \text{definition of power} \\
(a, b) \in R^n \circ R \\
\downarrow \quad \text{definition of composition} \\
\exists x : (a, x) \in R \land (x, b) \in R^n \\
\downarrow \quad \text{inductive hypothesis} \ R^n \subseteq R \\
\exists x : (a, x) \in R \land (x, b) \in R \\
\downarrow \quad R \text{ is transitive} \\
(a, b) \in R
\]

End of Proof