Induction and Recursion
(Sections 4.1-4.3)
[Section 4.4 optional]

Based on Rosen and slides by K. Busch

Induction

Induction is a very useful proof technique

In computer science, induction is used
to prove properties of algorithms

Induction and recursion are closely related

- Recursion is a description method for algorithms
- Induction is a proof method suitable for recursive algorithms

Use induction to prove that
a proposition $P(n)$ is true:

Inductive Basis: Prove that $P(1)$ is true

Inductive Hypothesis: Assume $P(k)$ is true
(for any positive integer k)

Inductive Step: Prove that $P(k+1)$ is true

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Inductive Hypothesis: Assume $P(k)$ is true
(for any positive integer k)

Inductive Step: Prove that $P(k+1)$ is true

In other words in inductive step we prove:

$$P(k) \rightarrow P(k+1)$$

for every positive integer k

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Inductive basis

$$P(1)$$

True

Inductive Step

$$P(k) \rightarrow P(k+1)$$

True

Proposition true for all positive integers

$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots$$

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Induction as a rule of inference:

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

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Theorem: $P(n): \quad 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

Proof:

Inductive Basis: $P(1): \quad 1 = \dfrac{1(1+1)}{2}$

Inductive Hypothesis: assume that it holds

$$P(k): \quad 1+2+\cdots k = \frac{k(k+1)}{2}$$

Inductive Step: We will prove

$$P(k+1): \quad 1+2+\cdots k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

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Inductive Step:

$$P(k+1): \quad 1+2+\cdots k+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

(inductive hypothesis)

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

End of Proof

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Harmonic numbers

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j}$$

$$j = 1, 2, 3, \ldots$$

Example: $$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

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Theorem: $$H_{2^n} \geq 1 + \frac{n}{2} \qquad n \geq 0$$

Proof:

Inductive Basis: $n = 0$

$$H_{2^n} = H_{2^0} = H_1 = 1 = 1 + \frac{0}{2} = 1 + \frac{n}{2}$$

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Inductive Hypothesis: $n = k$

Suppose it holds: $H_{2^k} \geq 1 + \frac{k}{2}$

Inductive Step: $n = k + 1$

We will show: $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \left(1 + \frac{k}{2}\right) + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$

from inductive hypothesis

End of Proof

Theorem: $H_{2^n} \leq 1+n$ $\qquad n \geq 0$

Proof:

Inductive Basis: $n=0$

$$H_{2^n} = H_{2^0} = H_1 = 1 = 1+0 = 1+n$$

Inductive Hypothesis: $n=k$

Suppose it holds: $H_{2^k} \leq 1+k$

Inductive Step: $n=k+1$

We will show: $H_{2^{k+1}} \leq 1+(k+1)$

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

$$= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

$$\leq (1+k) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

from inductive hypothesis

$$\leq (1+k) + 2^k \cdot \frac{1}{2^k+1}$$

$$\leq (1+k) + 1$$

$$= 1 + (k+1)$$

End of Proof

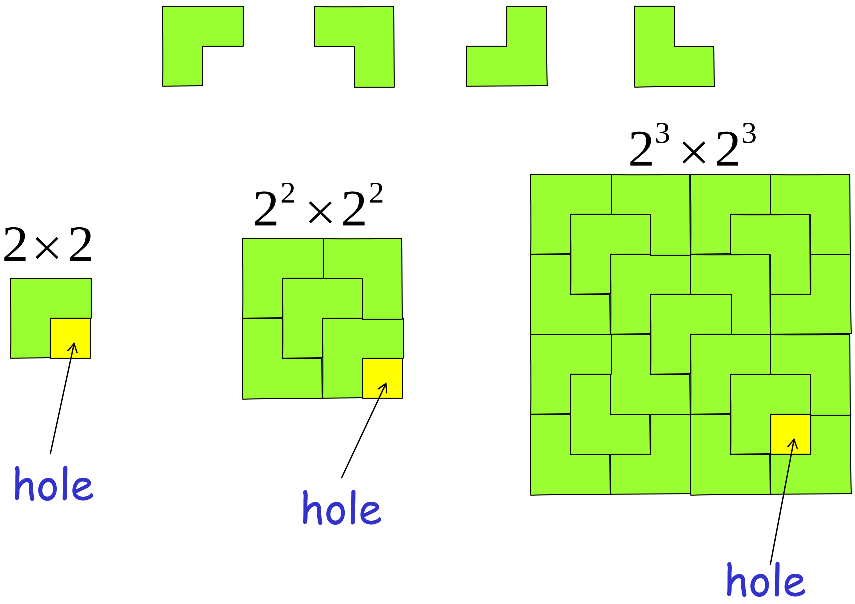
We have shown: $1 + \frac{n}{2} \leq H_{2^n} \leq 1 + n$

$$1 + \frac{\lfloor \log k \rfloor}{2} \leq H_k \leq 1 + \lceil \log k \rceil$$

$$H_k \approx \log k$$

(for large k)

Triominos



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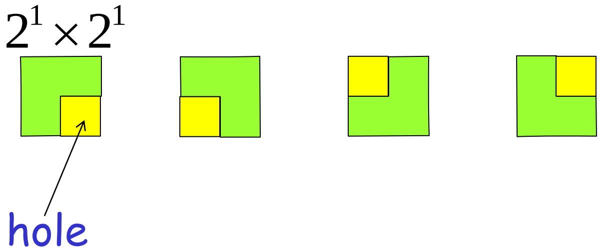
Theorem: Every $2^n \times 2^n$, $n \geq 1$ checkerboard with one square removed can be tiled with triominoes

Proof:

Inductive Basis: $n = 1$



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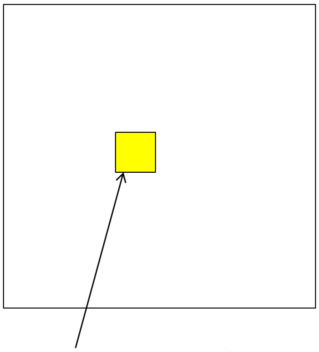
Inductive Hypothesis: $n = k$

Assume that a $2^k \times 2^k$ checkerboard can be tiled with the hole anywhere

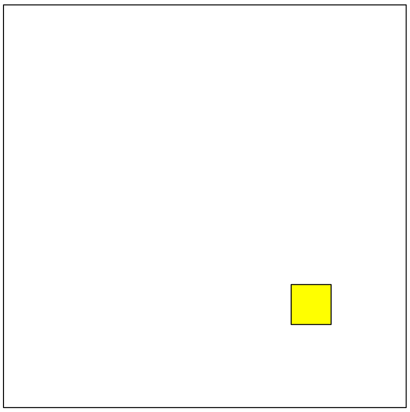
$2^k \times 2^k$



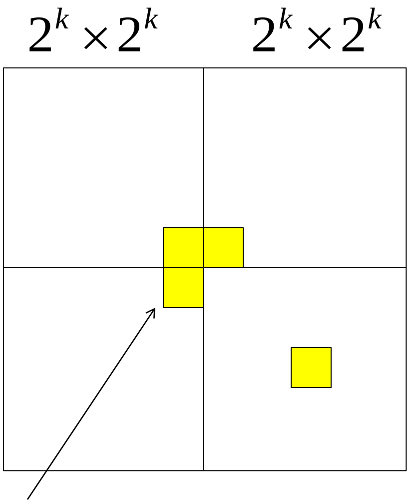
Hole can be anywhere

Inductive Step: $n = k + 1$

$2^{k+1} \times 2^{k+1}$

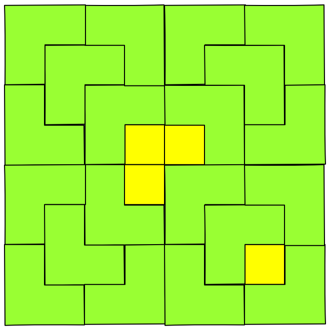


By inductive hypothesis $2^k \times 2^k$ squares with a hole can be tiled

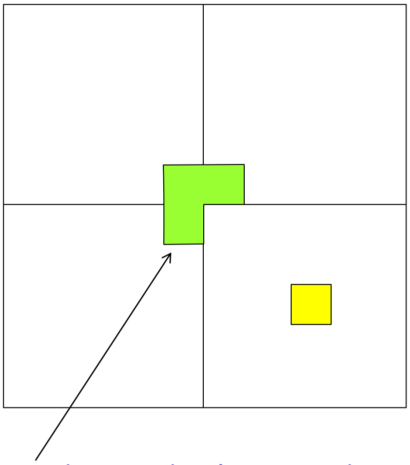


add three artificial holes

$2^3 \times 2^3$ case:

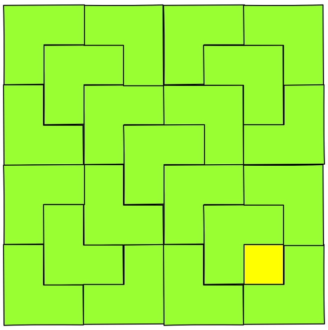


$2^k \times 2^k$ $2^k \times 2^k$



Replace the three holes with a triomino
Now, the whole area can be tiled

2^3 x 2^3 case:



End of Proof

Strong Induction

To prove $P(n)$:

Inductive Basis: Prove that $P(1)$ is true

Inductive Hypothesis:

Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ is true

Inductive Step: Prove that $P(k+1)$ is true

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Theorem: Every integer $n \geq 2$
is a product of primes
(Fundamental Theorem of Arithmetic)

Proof: (Strong Induction)

Inductive Basis: $n = 2$

Number 2 is a prime

Inductive Hypothesis: $2 \leq n \leq k$

Suppose that every integer between
2 and k is a product of primes

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Inductive Step: $n = k + 1$

If $k+1$ is prime then the proof is finished

If $k+1$ is not a prime then it is composite:

$$k+1 = a \cdot b \qquad 2 \le a, b \le k$$

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$$k+1 = a \cdot b \qquad 2 \le a, b \le k$$

By the inductive hypothesis:

$$\left.\begin{array}{c} i, j \ge 1 \\ a = p_1 p_2 \cdots p_i \\ b = q_1 q_2 \cdots q_j \\ \text{primes} \end{array}\right\} \Rightarrow k+1 = a \cdot b = \underbrace{p_1 \cdots p_i q_1 \cdots q_j}_{\text{primes}}$$

End of Proof

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Theorem: Every postage amount $n \geq 12$ can be generated by using 4-cent and 5-cent stamps

Proof: (Strong Induction)

Inductive Basis: We examine four cases (because of the inductive step)

$n = 12 = 4 + 4 + 4$

$n = 13 = 4 + 4 + 5$

$n = 14 = 5 + 5 + 4$

$n = 15 = 5 + 5 + 5$

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Inductive Hypothesis: $12 \leq n \leq k$

Assume that every postage amount between 12 and k can be generated by using 4-cent and 5-cent stamps

$$n = a \cdot 4 + b \cdot 5$$

Inductive Step: $n = k + 1$

If $12 \leq k \leq 14$ then the inductive step follows directly from inductive basis

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Consider: $k \geq 15$ $\qquad k+1=(k-3)+4$

$$12 \leq (k-3) \leq k$$

Inductive hypothesis

$$(k-3)=a' \cdot 4+b' \cdot 5$$

$$k+1=(k-3)+4=(a'+1) \cdot 4+b' \cdot 5$$

End of Proof

Recursion

Recursion is used to describe functions, sets, algorithms

Example: Factorial function $f(n)=n!$

Recursive Basis: $f(0)=1$

Recursive Step: $f(n+1)=(n+1) \cdot f(n)$

Recursive algorithm for factorial

```
factorial( n ) {
    if n = 1 then        //recursive basis
        return 1
    else                 //recursive step
        return n · factorial(n-1)
}
```

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Fibonacci numbers

$$f_0, f_1, f_2, f_3, \dots$$

Recursive Basis: $f_0 = 0, \quad f_1 = 1$

Recursive Step: $f_n = f_{n-1} + f_{n-2}$

$$n = 2,3,4,\dots$$

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$$f_0 = 0$$
$$f_1 = 1$$
$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$
$$f_7 = f_6 + f_5 = 8 + 5 = 13$$
$$\vdots$$

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Recursive algorithm for Fibonacci function

```
fibonacci(n) {
  if n ∈ {0,1} then    //recursive basis
    return n
  else                 //recursive step
    return fibonacci(n-1) + fibonacci(n-2)
}
```

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Iterative algorithm for Fibonacci function

fibonacci(n) {
  if $n = 0$ then $y \leftarrow 0$
  else {
    $x \leftarrow 0$
    $y \leftarrow 1$
    for $i \leftarrow 1$ to $n-1$ do {
      $z \leftarrow x + y$
      $x \leftarrow y$
      $y \leftarrow z$
    }
  return y
}

Theorem: $f_n > \delta^{n-2}$ for $n \geq 3$

$$\delta = \frac{1+\sqrt{5}}{2}$$ (golden ratio)

Proof: Proof by (strong) induction

Inductive Basis: $n = 3$ $n = 4$

$$f_3 = 2 > \delta$$

$$f_4 = 3 > \delta^2$$

Inductive Hypothesis: $3 \le n \le k$

Suppose it holds $f_n > \delta^{n-2}$

Inductive Step: $n = k+1$

We will prove $f_{k+1} > \delta^{(k-1)}$ for $4 \le k$

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δ is the solution to equation $x^2 - x - 1 = 0$

$$\Downarrow$$

$$\delta^2 = \delta + 1$$

$$\Downarrow$$

$$\delta^{k-1} = \delta^2 \cdot \delta^{k-3} = (\delta+1)\delta^{k-3} = \delta^{k-2} + \delta^{k-3}$$

$$\Downarrow$$

$$f_{k+1} = f_k + f_{k-1} \ge \delta^{k-2} + \delta^{k-3} = \delta^{k-1}$$

induction hypothesis

End of Proof

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Euclidean Algorithm for Greatest Common Divisor

Recursive Basis: $\gcd(a,0) = a$

Recursive Step: $\gcd(a,b) = \gcd(b, a \bmod b)$

$$a > b$$

Recursive Euclidean algorithm for greatest common divisor

```
gcd( a,b ) {        //assume a>b
   if b = 0 then         //recursive basis
      return a
   else                  //recursive step
      return gcd(b, a mod b)
}
```

Algorithm Mergesort

8 2 4 6 9 7 10 1 5 3

split

8 2 4 6 9   7 10 1 5 3

sort   sort

2 4 6 8 9   1 3 5 7 10

merge

1 2 3 4 5 6 7 8 9 10

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---

sort($a_1, a_2, \ldots, a_n$) {
    if $n > 1$ then {
        $m = \lfloor n/2 \rfloor$
        $A \leftarrow \mathrm{sort}(a_1, a_2, \ldots, a_m)$
        $B \leftarrow \mathrm{sort}(a_m, a_{m+1}, \ldots, a_n)$
        return $\mathrm{merge}(A, B)$
    }
    else return a_1
}

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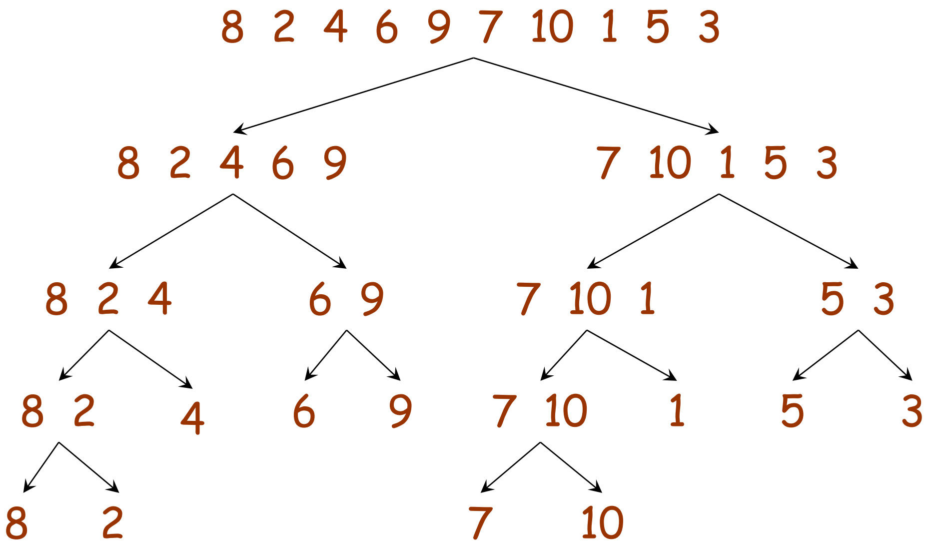
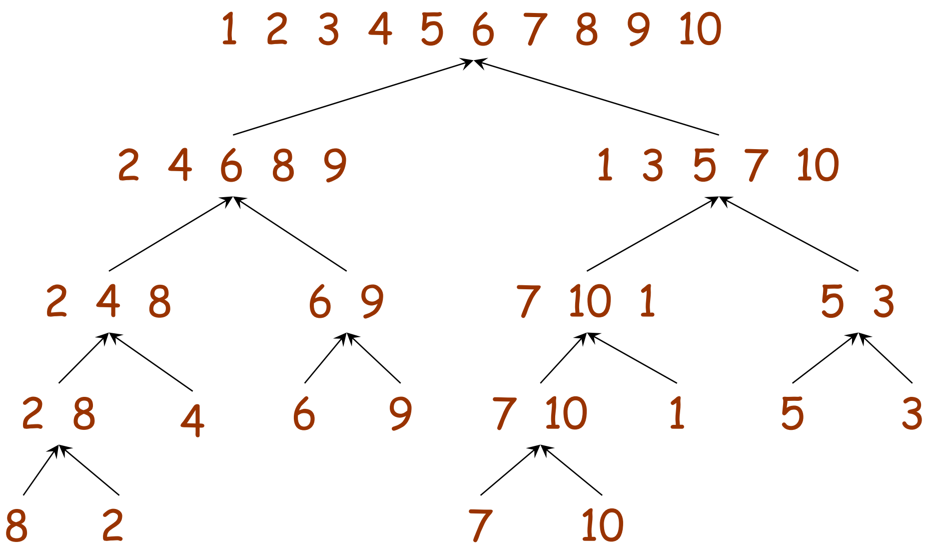
Input values of recursive calls



Input and output values of merging



merge(A, B) { //two sorted lists

$L \leftarrow \varnothing$

while $A \neq \varnothing$ and $B \neq \varnothing$ do {

Remove smaller first element of A, B from its list and insert it to L

}

if $A \neq \varnothing$ or $B \neq \varnothing$ then {

append remaining elements to L

}

return L

}

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merging 2 4 6 8 9

2 4 8   6 9

A	B	L	Comparison
2 4 8	6 9	2	2<6
4 8	6 9	2 4	4<6
8	6 9	2 4 6	6<8
8	9	3 4 6 8	8<9
	9	2 4 6 8 9	

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The total number of comparisons to merge two lists A, B is at most:

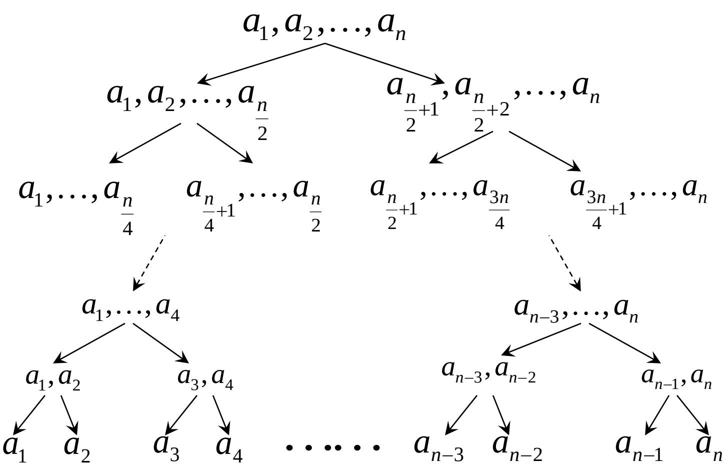
Merged size

$$\#\text{comparisons} \leq |A| + |B|$$

Length of A

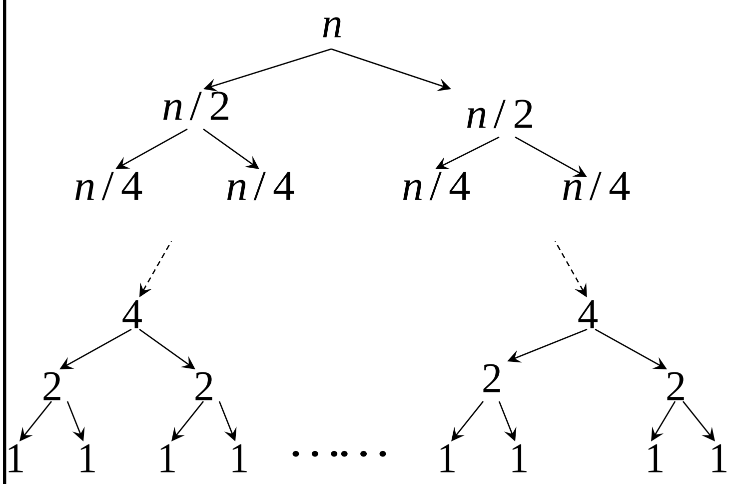
Length of B

Recursive invocation tree



Assume
$n = 2^k$

Recursive invocation tree



Elements per list

$n = 2^{\log n - 0}$

$n/2 = 2^{\log n - 1}$

$n/4 = 2^{\log n - 2}$

$\vdots$

$4 = 2^{\log n - (\log n - 2)}$

$2 = 2^{\log n - (\log n - 1)}$

$1 = 2^{\log n - \log n}$

Assume $n = 2^k$

#levels of tree $= 1 + \log n$

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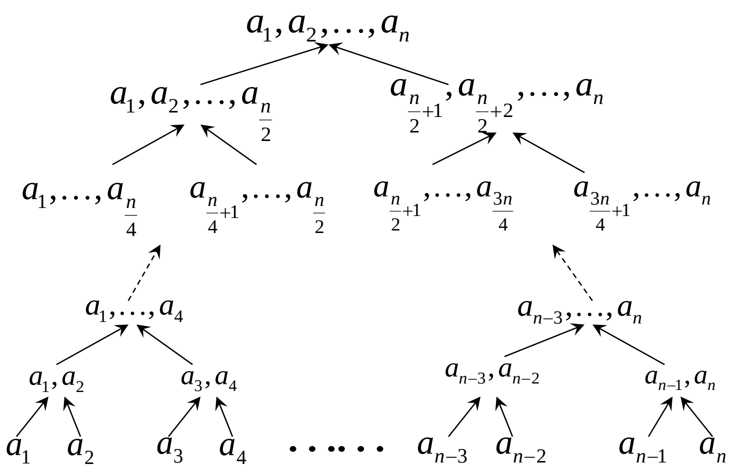
merging tree



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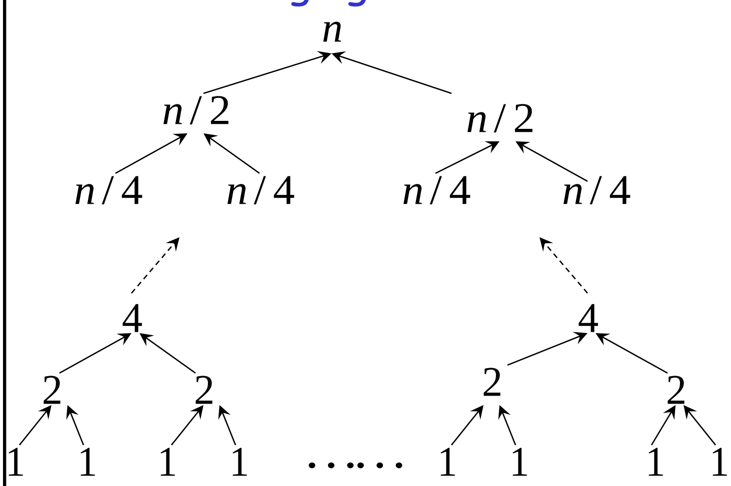
merging tree



Elements per list

n

$n/2$

$n/4$

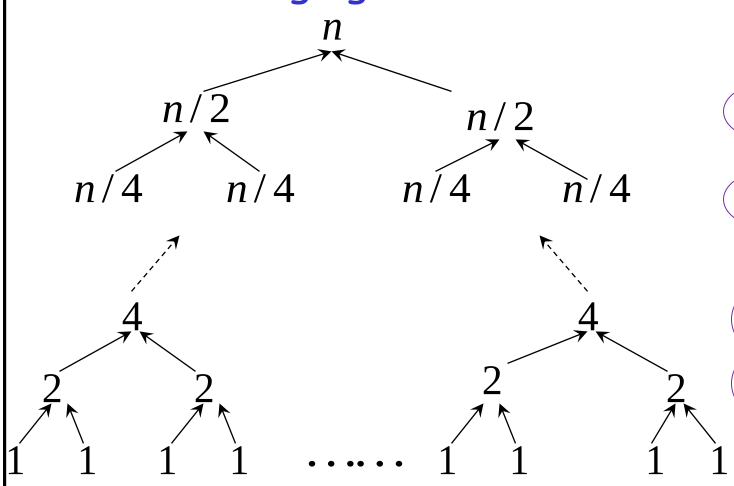
$\vdots$

4

2

1

merging tree



Comparisons per level

$n \cdot 1 = n$

$n/2 \cdot 2 = n$

$n/4 \cdot 4 = n$

$\vdots$

$4 \cdot n/4 = n$

$2 \cdot n/2 = n$

Merges per level

Elements per merge

Total cost: $n \cdot (\#\text{levels} - 1) = n \log n$

If $n = 2^k$ the number of comparisons is at most $n \log n$

If $n \neq 2^k$ the number of comparisons is at most $m \log m < 2n \log 2n < c \cdot n \log n$

where $m = 2^{\lceil \log n \rceil} < 2n$

Therefore, worst-case running time of merge sort is $\approx n \log n$

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