Announcements

- Reading
  - 7th edition: 2.5 (Cardinality) + p. 201 and 13.5
- Homework 10 out today, due next Friday
  - Homework 9 due today

Last lecture highlights

- Sequential Circuits for FSMs
  - Combinational logic for transition function
  - Sequential logic for iteration
- Carry-look-ahead Adders
  - $C_4 = G_4 + G_3 P_4 + G_2 P_3 P_4 + G_1 P_2 P_3 P_4 + G_0 P_1 P_2 P_3 P_4$ etc.
- Composition trees and Parallel Prefix

Computing & Mathematics

Computers as we know them grew out of a desire to avoid bugs in mathematical reasoning

A Brief History of Reasoning

- Ancient Greece
  - Deductive logic
    - Euclid’s Elements
  - Infinite things are a problem
    - Zeno’s paradox
- 1670's-1800's Calculus & infinite series
  - Suddenly infinite stuff really matters
  - Reasoning about infinite still a problem
    - Tendency for buggy or hazy proofs
- Mid-late 1800's
  - Formal mathematical logic
    - Boole Boolean Algebra
  - Theory of infinite sets and cardinality
    - Cantor
    “There are more real #’s than rational #’s”
A Brief History of Reasoning

- **1900**
  - Hilbert’s famous speech outlines goal: mechanize all of mathematics
  - 23 problems

- **1930’s**
  - Gödel, Turing show that Hilbert’s program is impossible.
    - Gödel’s Incompleteness Theorem
    - Undecidability of the Halting Problem
  - Both use ideas from Cantor’s proof about reals & rationals

Turing Machines

**Church-Turing Thesis**

Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

- *Evidence*
  - Huge numbers of equivalent models to TM’s based on radically different ideas

Cardinality

**Def:** Two sets A and B are the same size (same cardinality) iff there is a 1-1 and onto function f:A→B

- Also applies to infinite sets

Starting with Cantor

- **1930’s**
  - How can we formalize what algorithms are possible?
    - Turing machines (Turing, Post)
      - basis of modern computers
    - Lambda Calculus (Church)
      - basis for functional programming
      - All are equivalent!
    - μ-recursive functions (Kleene)
      - alternative functional programming basis

Cardinality

- The natural numbers and even natural numbers have the same cardinality:
  - \[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...\]
  - \[0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, ...\]
  - n is matched with 2n
Countability

Definition: A set is countable iff it is the same size as some subset of the natural numbers

Equivalent: A set S is countable iff there is an onto function \( g: \mathbb{N} \rightarrow S \)

Equivalent: A set S is countable iff we can write \( S = \{s_1, s_2, s_3, \ldots \} \)

The set of all integers is countable

Is the set of positive rational numbers countable?

• We can’t do the same thing we did for the integers
  – Between any two rational numbers there are an infinite number of others

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Positive Rational Numbers

1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 \ldots
2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 \ldots
3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 \ldots
4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 \ldots
5/1 5/2 5/3 5/4 5/5 5/6 5/7 \ldots
6/1 6/2 6/3 6/4 6/5 6/6 \ldots
7/1 7/2 7/3 7/4 7/5 \ldots
\ldots \ldots \ldots \ldots

\{Positive Rational Numbers\} is Countable

1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 \ldots
2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 \ldots
3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 \ldots
4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 \ldots
5/1 5/2 5/3 5/4 5/5 5/6 5/7 \ldots
6/1 6/2 6/3 6/4 6/5 6/6 \ldots
7/1 7/2 7/3 7/4 7/5 \ldots
\ldots \ldots \ldots \ldots

\{Positive Rational Numbers\} is Countable

\( \mathbb{Q}^+ = \{1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 2/3, 3/2, 1/4, 5/1, 4/2, 3/3, 2/4, 1/5, \ldots\} \)

List elements in order of
  – numerator + denominator
  – breaking ties according to denominator
    * Only k numbers when the total is k

Technique is called “dovetailing”
Claim: $\Sigma^*$ is countable for every finite $\Sigma$

The set of all Java programs is countable

What about the Real Numbers?

Q: Is every set is countable?

A: Theorem [Cantor] The set of real numbers (even just between 0 and 1) is NOT countable

Proof is by contradiction using a new method called “diagonalization”

Real numbers between 0 and 1: $\mathbb{R}^{(0,1)}$

• Every number between 0 and 1 has an infinite decimal expansion:
  
  \[
  \begin{align*}
  1/2 &= 0.50000000000000000000000000000000...
  \\
  1/3 &= 0.33333333333333333333333333333333...
  \\
  1/7 &= 0.14285714285714285714285714285714...
  \\
  \pi - 3 &= 0.14159265358979323846264...
  \\
  1/5 &= 0.199999999999999999999999999999999999...
  \\
  &= 0.20000000000000000000000000000000...
  \end{align*}
  \]

Proof by contradiction

• Suppose that $\mathbb{R}^{(0,1)}$ is countable
• Then there is some listing of all elements $\mathbb{R}^{(0,1)} = \{ r_1, r_2, r_3, r_4, \ldots \}$
• We will prove that in such a listing there must be at least one missing element which contradicts statement "$\mathbb{R}^{(0,1)}$ is countable"
• The missing element will be found by looking at the decimal expansions of $r_1, r_2, r_3, r_4, \ldots$

Representations as decimals

Representation is unique except for the cases that decimal ends in all 0’s or all 9’s.

\[
\begin{align*}
  x &= 0.1999999999999999999999999999999999999999999999999999999999999999...
  \\
  10x &= 1.9999999999999999999999999999999999999999999999999999999999999999...
  \\
  9x &= 1.8 \text{ so } x = 0.20000000000000000000000000000000...
\end{align*}
\]

Won’t allow the representations ending in all 9’s

All other representations give elements of $\mathbb{R}^{(0,1)}$
Supposed Listing of $\mathbb{R}^{[0,1)}$

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Flipped Diagonal

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Supposed Listing of $\mathbb{R}^{[0,1)}$

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Flipped Diagonal Number $D$

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<td>$D = 0.$</td>
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<td>$D$ is in $\mathbb{R}^{[0,1)}$</td>
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<td>But for all $n$, we have</td>
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<td>$D_{10}$ since they differ on</td>
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<td>$n^{th}$ digit (which is not 0 or 9)</td>
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<td>$\Rightarrow \mathbb{R}^{[0,1)}$ is not countable</td>
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Non-computable functions

- We have seen that
  - The set of all (Java) programs is countable
  - The set of all functions $f : \mathbb{N} \rightarrow \{0,1,...,9\}$ is not countable

- So...
  - There must be some function $f : \mathbb{N} \rightarrow \{0,1,...,9\}$ that is not computable by any program!