1. Composing relations:

Recall: \( S \circ R = \{(a, c) \mid \exists b \text{ s.t.} (a, b) \in R \text{ and } (b, c) \in S\} \)

We define the following relations:

- \( (a, b) \in \text{Sibling}: \) \( b \) is \( a \)'s sibling
- \( (a, b) \in \text{Mother}: \) \( b \) is \( a \)'s mother
- \( (a, b) \in \text{Parent}: \) \( b \) is \( a \)'s parent
- \( (a, b) \in \text{Daughter}: \) \( b \) is \( a \)'s daughter
- \( (a, b) \in \text{Son}: \) \( b \) is \( a \)'s son
- \( (a, b) \in \text{Child}: \) \( b \) is \( a \)'s child

Use these relations to express the following:

(a) \( \{(a, c) \mid c \text{ is } a \text{'s niece}\} \): \( \text{Daughter} \circ \text{Sibling} \)
(b) \( \{(a, c) \mid c \text{ is } a \text{'s grandson}\} \): \( \text{Son} \circ \text{Child} \)
(c) \( \{(a, c) \mid c \text{ is } a \text{'s grandmother}\} \): \( \text{Mother} \circ \text{Parent} \)

2. Proving relationship properties

Prove that the relation \( R \) on a set \( A \) is symmetric if and only if \( R = R^{-1} \).

For an "if and only if" proof we need to prove both directions:

(a) "only if" direction: Prove that if \( R \) is symmetric, then \( R = R^{-1} \)

Assume that \( R \) is symmetric.

To show that \( R = R^{-1} \), we must show both directions:

- Show that \( R \subseteq R^{-1} \)
  Let \( (x, y) \) be an arbitrary member of \( R \). Then:
  \( (y, x) \in R \) because \( R \) is symmetric
  \( (x, y) \in R^{-1} \) by definition of inverse
  \( \text{QED} \)

- Show that \( R^{-1} \subseteq R \)
  Let \( (x, y) \) be an arbitrary member of \( R^{-1} \). Then:
  \( (y, x) \in R \) by definition of inverse
  \( (x, y) \in R \) because \( R \) is symmetric
  \( \text{QED} \)

We have shown by direct proof that if \( R \) is symmetric then \( R = R^{-1} \).
(b) “if” direction: Prove that if $R = R^{-1}$, then $R$ is symmetric

Assume $R = R^{-1}$. To show that $R$ is symmetric, we must show that for any arbitrary $(x, y)$ in $R$, $(y, x)$ is also in $R$.

Let $(x, y)$ be an arbitrary member of $R$.

$(y, x) \in R^{-1}$ by definition of inverse

$(y, x) \in R$ by assumption that $R = R^{-1}$

QED

We have shown by direct proof that if $R = R^{-1}$, then $R$ is symmetric.

We have shown both the "if" and "only if" directions. Therefore, we have proven that the relation $R$ on a set $A$ is symmetric if and only if $R = R^{-1}$. 