October 12, 2011

   - Interactive Demonstration Applets
     - Truth Tables
     - Equivalences
   - Self Assessments
     - Conditional Statements
     - Quantified Statements
   - Guide to Writing Proofs
   - Common Mistakes

2. Prove that \((p \rightarrow r) \land (q \rightarrow r) \equiv (p \lor q) \rightarrow r\) by rewriting with equivalences.

\[
\begin{align*}
(p \rightarrow r) \land (q \rightarrow r) & \equiv (p \lor q) \rightarrow r \\
& \equiv \neg(p \lor q) \lor r \quad \text{Law of implication} \\
& \equiv (\neg p \land \neg q) \lor r \quad \text{DeMorgan’s} \\
& \equiv (\neg p \lor r) \land (\neg q \lor r) \quad \text{Distributive} \\
& \equiv (p \rightarrow r) \land (q \rightarrow r) \quad \text{Law of implication}
\end{align*}
\]

3. Prove that \((p \land q) \rightarrow (p \rightarrow q)\) is a tautology by rewriting with equivalences.

\[
\begin{align*}
(p \land q) \rightarrow (p \rightarrow q) & \equiv T \\
\neg(p \land q) \lor (p \rightarrow q) & \quad \text{Law of implication} \\
\neg(p \land q) \lor (\neg p \lor q) & \quad \text{Law of implication} \\
(\neg p \lor \neg q) \lor (\neg p \lor q) & \quad \text{DeMorgan} \\
\neg p \lor \neg q \lor \neg p \lor q & \quad \text{Associative} \\
\neg p \lor \neg q \lor q \lor \neg p & \quad \text{Commutative} \\
\neg p \lor T \lor \neg p & \quad \text{Negation} \\
\neg p \lor T & \quad \text{Domination} \\
T & \quad \text{Domination}
\end{align*}
\]
4. Find the values, if any, of the Boolean variable $x$ that satisfies these equations:

(a) $x \cdot 1 = 0$  0
(b) $x + x = 0$  0
(c) $x \cdot 1 = x$  0, 1
(d) $x \cdot \bar{x} = 1$  none

5. Use truth tables to express the values of these Boolean functions:

(a) $F(x, y, z) = xy + xz$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$xy$</th>
<th>$xz$</th>
<th>$xy + xz$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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(b) $F(x, y, z) = \overline{y}(xz + \bar{x}\bar{z})$

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<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$\overline{y}$</th>
<th>$xz$</th>
<th>$\bar{x}\bar{z}$</th>
<th>$\overline{y}(xz + \bar{x}\bar{z})$</th>
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6. For a Boolean function on each of the following number of inputs:

- How many rows are in the truth table?
- How many different Boolean functions are possible?
  - 3 inputs ("a Boolean function of degree 3")  8 rows; $2^8$ functions
  - 4 inputs  16 rows; $2^{16}$ functions
  - 30 inputs  $2^{30}$ rows; $2^{(2^{30})}$ (about $2^{4 \text{billion}}$) functions

In general for $n$ variables there are $2^n$ rows and $2^{(2^n)}$ possible functions.

7. Half adder

(a) Write the truth table for a half adder (takes two bits, $x$ and $y$, and outputs two bits - $s$ (sum) and $c$ (carry):
(b) Use the truth table to write the boolean expressions for outputs \( s \) and \( c \). (Don’t minimize.)

\[
\begin{array}{c|c|c|c}
 x & y & s & c \\
 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 \\
 1 & 0 & 1 & 0 \\
 1 & 1 & 0 & 1 \\
\end{array}
\]

\( s = \bar{x}y + x\bar{y} \)

\( c = xy \)

(c) How many gates will you need in a circuit that implements these expressions? \( 6 \text{ gates: 3 AND, 1 OR, 2 NOT} \)

(d) Draw the circuit.

(e) Minimize the expression for \( s \). Now how many gates do you need?

\( s = (x + y)\bar{xy}; \quad 4 \text{ gates: 2 AND, 1 OR, 1 NOT} \) (Notice that we can reuse the \( xy \) AND gate.)

(f) Draw the simplified circuit.

Note: All of the above was done with just AND, OR and NOT gates. If we allow XOR gates, then we can have a much simpler circuit with just 2 gates (1 XOR, 1 AND):
\[ c = xy \]
\[ s = x \oplus y \]

\[ \begin{array}{ccccc}
\hline
x & y & c \\
\hline
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\hline
\end{array} \]

8. Repeat the steps from the above problem (using \( t \) as the single output value) for the Boolean function given by the following truth table:

\[ \begin{array}{cccccc}
\hline
x & y & z & t \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\hline
\end{array} \]

(a) Use the truth table to write the boolean expression for \( t \):
\[ t = xyz + \overline{xyz} \]

(b) How many gates would you need for this circuit?
4: 2 AND, 1 OR, 1 NOT

(c) Draw the circuit:

\[ \begin{array}{ccccc}
\hline
x & \overline{y} & AND & OR & t \\
\hline
y & z & \overline{z} & & \\
\hline
\end{array} \]

(d) Minimize the expression for \( t \). Now how many do you need?:
\[ t = yz \] just 1 ADD gate

(e) Draw the simplified circuit:

\[ \begin{array}{ccccc}
\hline
y & AND & t \\
\hline
z & \overline{z} & & \\
\hline
\end{array} \]